EBA2911 Mathematics for Business Analytics autumn 2019

Exercises

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

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Lecture 16

Repetition.

Multiple choice exam 2019 spring
Multiple choice exam 2018 autumn

Problems for the exercise session Wednesday 20 Nov. from 14 o'clock in B2-085 and Monday 9 Dec. from 14 o'clock in the Study Area

Problem 1

a) Determine the equation of the ellipse in figure 1. Also determine the centre and the semi-axes for the ellipse.

b) Determine the function expression of the hyperbola in 1 (with asymptotes drawn). Also determine the equations for the asymptotes of the hyperbola.

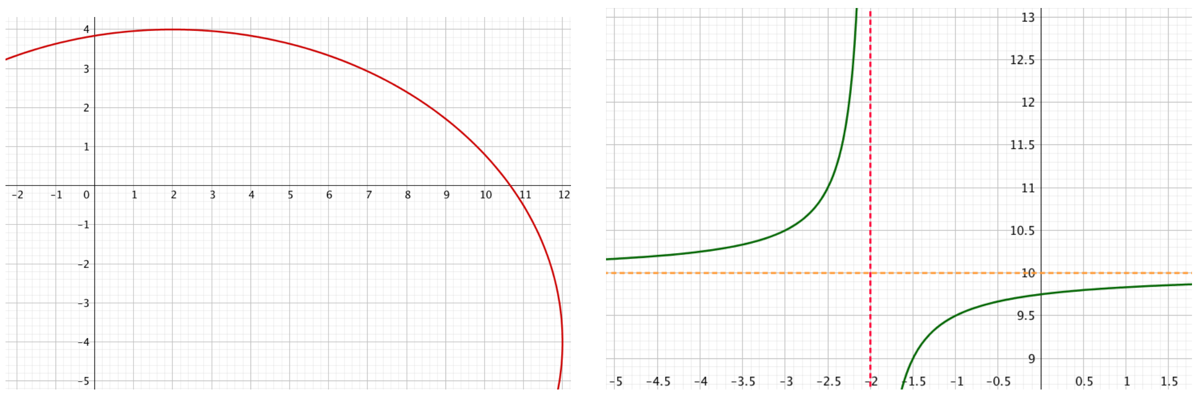

Figure 1: An ellipse and a hyperbola

Problem 2

a) A curve is given by the solutions of the equation $64x^2 + 100y^2 - 256x + 800y = 4544$. Use implicit differentiation to express y' in terms of y and x.

b) Determine the points on the curve with $x = 8$.

c) Determine the tangent equations of the points in (b).

Problem 3 Determine the inverse function expression $g(x)$ and the domain of definition D_g of the function $f(x)$ with domain of definition D_f.

a) $f(x) = 2x + 5$ with $D_f = [3, \infty\rangle$ b) $f(x) = 10 + \frac{1}{x-2}$ with $D_f = \langle 2, \infty\rangle$

c) $f(x) = (x-5)^3 + 2$ with $D_f = \mathbb{R}$ (all real numbers)

d)

$$f(x) = \begin{cases} \frac{18}{x} & \text{if } 0 < x \leqslant 6 \\ 4 - \frac{x}{6} & \text{if } 6 < x \leqslant 45 \end{cases}$$

Problem 4 Determine exact and approximate values in the following problems. We assume continuous compounding.

a) You deposit 50 000 into an account. Determine the nominal interest which gives you a balance of 150 000 after 15 years.
b) Determine the nominal interest such that the present value of 9 million in 6 years from now is 5 million.
c) You deposit 500 000 into an account with 3,9% nominal interest. Determine how long time it takes before the balance is 1 200 000.
d) You consider an investment of 45 million in a project which promises a single payment of 70 million. Suppose the internal rate of return is 10%. Determine when the payment should happen if the deal is balanced (fair).

Problem 5 Solve the inequalities.
a) $3e^x \leqslant 10$ b) $\ln(x-7) > 5$ c) $\ln \frac{2x+5}{x-3} < 0$ d) $\frac{e^x}{e^x-3} < -2$

Problem 6 We have the function $f(x) = x^3 - 4x^2 + 4x + 3$ with domain of definition $D_f = [0,3]$.
a) Determine the stationary points of $f(x)$
b) Determine where $f(x)$ is growing and where $f(x)$ is decreasing
c) Determine the local minimum and maximum points of $f(x)$
d) Determine how many global minimum and maximum points $f(x)$ has and compute maximum and minimum of the function.
e) Determine where $f(x)$ is convex/concave.

Problem 7 Determine the stationary points of $f(x)$. Determine where $f(x)$ is strictly decreasing/increasing. Determine possible local minimum and maximum points.
a) $f(x) = 5 - \ln(x^2 - 10x + 30)$ b) $f(x) = e^{x^3-12x}$

Problem 8 We have a function $f(x) = \ln(-0{,}01x^2 + 0{,}8x - 12)$ with domain of definition $D_f = \langle 20, 60\rangle$. Determine (local) minimum and maximum points. Explain why the stationary points gives minimum/maximum by applying convexity/concavity of the function. Compute maximum and minimum of the function.

Solutions

Note: There are alternative solutions for many of the problems which are (at least) as good as the ones I have chosen here.

Problem 1

a) The centre of the ellipse is the intersection point of the symmetry lines $x = 2$ and $y = -4$, that is $\underline{\underline{(2,-4)}}$. We read off the horizontal semi-axis: $\underline{\underline{10}}$, vertical semi-axis: $\underline{\underline{8}}$. Standard form for the equation of the ellipse is hence $\underline{\underline{\frac{(x-2)^2}{100} + \frac{(y+4)^2}{64} = 1}}$.
b) We read off the vertical asymptote: the line $x = -2$, horizontal asymptote: the line $y = 10$. The standard form of the function expression for the hyperbola is then $f(x) = 10 + \frac{a}{x+2}$. We determine a by inserting a point on the graph: $(-3, 10{,}5)$. It gives an equation for a: $10 + \frac{a}{-3+2} = 10{,}5$ which has the solution $a = -0{,}5$. Hence $f(x) = \underline{\underline{10 - \frac{0{,}5}{x+2}}}$

Problem 2

a) We differentiate both sides of the equation by applying the power rule several times and the chain rule on y^2 [we think of y as a function of x locally on the curve, i.e. $y = y(x)$, but we don't need any expression for $y(x)$]. Then we get $128x + 200y \cdot y' - 256 + 800y' = 0$ which gives the equation $200(y+4)y' = 128(2-x)$, that is $\underline{\underline{y' = 0{,}64 \cdot \frac{2-x}{y+4}}}$
b) With $x = 8$ we get the equation $64^2 + 100y^2 - 256 \cdot 8 + 800y = 4544$, i.e. $y^2 + 8y = 24{,}96$. We complete the square: $(y+4)^2 = 24{,}96 + 16 = 40{,}96$. It gives $y = -4 \pm 6{,}4$, i.e. $y = -10{,}4$ or $y = 2{,}4$ which gives the points $\underline{\underline{P = (8,\ -10{,}4)}}$ and $\underline{\underline{Q = (8,\ 2{,}4)}}$

c) For P: We insert $x = 8$ and $y = -10{,}4$ into the expression for y' from (a): $y' = 0{,}64 \cdot \frac{2-8}{-10{,}4+4} = 0{,}6$. The point-slope formula gives $p(x) - (-10{,}4) = 0{,}6(x-8)$, i.e. $\underline{\underline{p(x) = 0{,}6x - 15{,}2}}$ (exact answer)

For Q: $y' = 0{,}64 \cdot \frac{2-8}{2{,}4+4} = -0{,}6$. The point-slope formula gives $q(x) - 2{,}4 = -0{,}6(x-8)$, dvs $\underline{\underline{q(x) = -0{,}6x + 7{,}2}}$ (exact answer)

Note: The curve is given in Problem 1a, but you don't have to know this to solve Problem 2.

Problem 3

a) To determine the expression $g(x)$ for the inverse function we put $y = 2x+5$ and solve the equation for x. It gives $x = 0{,}5y - 2{,}5$. Then we change the variables and get $\underline{\underline{g(x) = 0{,}5x - 2{,}5}}$. As always with the inverse function $D_g = R_f$. To determine the range of $f(x)$ we see that $f(x)$ is an increasing (and linear) function with minimum value $f(3) = 11$ and which attains all larger numbers as x increases, that is $R_f = [11, \infty\rangle$ and hence $\underline{\underline{D_g = [11, \infty\rangle}}$

b) We solve the equation $y = 10 + \frac{1}{x-2}$ for x. Get $x = 2 + \frac{1}{y-10}$. Changes variables and get the expression $\underline{\underline{g(x) = 2 + \frac{1}{x-10}}}$ for the inverse function. $f(x)$ is a hyperbola with the line $x = 2$ as vertical asymptote, which is strictly decreasing for $x > 2$, and which has the line $y = 10$ as horizontal asymptote. Then $R_f = \langle 10, \infty\rangle$ and $\underline{\underline{D_g = \langle 10, \infty\rangle}}$

c) We solve the equation $y = (x-5)^3 + 2$ for x and get $x = 5 + (y-2)^{\frac{1}{3}}$. Then $\underline{\underline{g(x) = 5 + (x-2)^{\frac{1}{3}}}}$ is the expression for the inverse the function. Because $y = f(x)$ becomes as negative as you want by choosing sufficiently large negative x, and $y = f(x)$ becomes as positive as you want by choosing large positive x, we get $R_f = \mathbb{R}$. Hence $\underline{\underline{D_g = \mathbb{R}}}$

d) We consider $f(x)$ as consisting of two different functions with separate domains of definition and do as in (a-c) for each of them. It gives $\underline{\underline{g(x) = \begin{cases} \frac{18}{x} & \text{if } x \geqslant 3 \\ 24 - 6x & \text{if } -\frac{21}{6} \leqslant x < 3 \end{cases}}}$

Problem 4

a) Let r be the nominal interest. The annual growth factor is then e^r and we obtain the equation $50\,000 \cdot e^{15r} = 150\,000$ which gives the equation $e^{15r} = 3$. We put the left and right hand side and into $\ln(x)$ and get the equation $15r = \ln(3)$ with the solution $\underline{\underline{r = \frac{\ln 3}{15} = 7{,}32\%}}$

b) Let r be the nominal interest. The present value of 9 million is then (in millions) $\frac{9}{e^{6r}}$ which is assumed to be 5 million. We get the equation $\frac{9}{e^{6r}} = 5$, i.e. $e^{6r} = \frac{9}{5}$. Insert the left and the right hand side into $\ln(x)$ and get $6r = \ln(\frac{9}{5}) = \ln(9) - \ln(5)$, i.e $\underline{\underline{r = \frac{\ln 9 - \ln 5}{6} = 9{,}80\%}}$

c) We put $x =$ number of years the money has to be deposited. Then we get the equation $500\,000 \cdot e^{0{,}039x} = 1\,200\,000$, i.e $e^{0{,}039x} = \frac{12}{5}$. Insert the left and the right hand side into $\ln(x)$. We get that the money has to be deposited in $\underline{\underline{x = \frac{\ln 12 - \ln 5}{0{,}039} = 22{,}45}}$ years

d) We put $x =$ the number of years between the investment and the payment. The annual growth factor is 1,1 and hence we get the equation $45 \cdot 1{,}1^x = 70$, i.e. $1{,}1^x = \frac{70}{45}$. Insert the left and the right hand side into e^x and get $x \cdot \ln(1{,}1) = \ln 70 - \ln 45$. Hence the payment has to occur $\underline{\underline{\frac{\ln 70 - \ln 45}{\ln(1{,}1)} = 4{,}64}}$ years after the investment.

Problem 5

a) We insert the left and the right hand side of the inequality $e^x \leqslant \frac{10}{3}$ into $\ln(x)$ and get $\underline{\underline{x \leqslant \ln(10) - \ln(3)}}$

b) We insert the left and the right hand side of the inequality into e^x and get the inequality $x - 7 > e^5$ which gives $\underline{\underline{x > 7 + e^5}}$

c) We insert the left and the right hand side of the inequality into e^x and get the inequality $\frac{2x+5}{x-3} < 1$, i.e $\frac{2x+5}{x-3} - 1 < 0$, i.e. $\frac{x+8}{x-3} < 0$. Now we use a sign diagram and get $\underline{\underline{x \in \langle -8, 3\rangle}}$

d) We can put $u = e^x$ and solve the inequality $\frac{u}{u-3} < -2$, i.e. $\frac{u}{u-3} + 2 < 0$, i.e. $\frac{3(u-2)}{u-3} < 0$. Now we use a sign diagram and get $e^x = u \in \langle 2, 3\rangle$, that is $\underline{\underline{x \in \langle \ln(2), \ln(3)\rangle}}$.

Problem 6

a) The stationary points are the solutions of the equation $f'(x) = 0$ with $0 \leqslant x \leqslant 3$. We compute $f'(x) = 3x^2 - 8x + 4$. The equation $3x^2 - 8x + 4 = 0$ has the solutions $\underline{\underline{x = \frac{2}{3}}}$, $\underline{\underline{x = 2}}$ which both are contained in D_f.

b) We have $f'(x) > 0$ for $x < \frac{2}{3}$ and for $x > 2$, and $f'(x) < 0$ for $\frac{2}{3} < x < 2$. Then $f(x)$ is increasing for x in $\underline{\underline{[0, \frac{2}{3}]}}$, decreasing for x in $\underline{\underline{[\frac{2}{3}, 2]}}$ and increasing for x in $\underline{\underline{[2, 3]}}$.

c) Since there are no cusp points for $f(x)$, the local minimum/maximum points of $f(x)$ are either stationary points or end points. From (6b) we get that $\underline{\underline{x = 0}}$ and $\underline{\underline{x = 2}}$ are local minimum points while $\underline{\underline{x = \frac{2}{3}}}$ and $\underline{\underline{x = 3}}$ are local maximum points.

d) To find the global extremal points we calculate $f(0) = 3$ and $f(2) = 3$ which hence gives that both $\underline{\underline{x = 0}}$ and $\underline{\underline{x = 2}}$ are global minimum points. Because $f(\frac{2}{3}) = 4{,}19$ while $f(3) = 6$, $\underline{\underline{x = 3}}$ is the only global maximum point. Minimum of the function is hence $f(0) = f(2) = \underline{\underline{3}}$ while maximum is $f(3) = \underline{\underline{6}}$.

e) We compute $f''(x) = 6x - 8$. $f(x)$ is concave for those x such that $f''(x) \leqslant 0$, i.e. $6x - 8 \leqslant 0$, i.e $\underline{\underline{x \leqslant \frac{4}{3}}}$. $f(x)$ is convex for those x such that $f''(x) \geqslant 0$, i.e. $\underline{\underline{x \geqslant \frac{4}{3}}}$.

Problem 7

a) We use the chain rule with $u(x) = x^2 - 10x + 30$, $g(u) = 5 - \ln(u)$, $u'(x) = 2x - 10 = 2(x-5)$, $g'(u) = -\frac{1}{u}$ which gives $f'(x) = -\frac{2(x-5)}{x^2-10x+30}$. Because $x = 5$ is not a root in the denominator the quotient cannot be simplified. We solve the equation $f'(x) = 0$, i.e. $2(x-5) = 0$, i.e. $\underline{\underline{x = 5}}$ is the only stationary point. We complete the square $x^2 - 10x + 30 = (x-5)^2 + 5$ which always is $\geqslant 5$. Hence we see that $f'(x)$ is larger than 0 for $x < 5$ and $f'(x)$ is smaller than 0 for $x > 5$. Hence $f(x)$ is strictly increasing in the interval $\underline{\underline{\langle -\infty, 5]}}$ and strictly decreasing in the interval $\underline{\underline{[5, \infty\rangle}}$ and $\underline{\underline{x = 5}}$ is a global maximum point.

b) We use the chain rule with $u(x) = x^3 - 12x$, $g(u) = 3e^u$, $u'(x) = 3x^2 - 12 = 3(x^2-4)$ and $g'(u) = 3e^u$ which gives $f'(x) = 3(x^2-4)e^{x^3-12x}$. Then we solve the equation $f'(x) = 0$, i.e. $3(x^2-4)e^{x^3-12x} = 0$. We have $3e^u > 0$ and hence we get the equation $x^2 - 4 = 0$, i.e. $\underline{\underline{x = \pm 2}}$ are the only stationary points of $f(x)$. We also get the factorisation $f'(x) = 3(x+2)(x-2)e^{x^3-12x}$. We then see (e.g. by use of a sign diagram) that $f'(x)$ is negative for x in $\langle -2, 2\rangle$ and positive for x in $\langle -\infty, -2\rangle \cup \langle 2, \infty\rangle$. Hence $f(x)$ is strictly increasing in the interval $\underline{\underline{\langle -\infty, -2]}}$ and in the interval $\underline{\underline{[2, \infty\rangle}}$, and is strictly decreasing in the interval $\underline{\underline{[-2, 2]}}$. Hence $\underline{\underline{x = -2}}$ is a local maximum point and $\underline{\underline{x = 2}}$ is a local minimum point.

Problem 8 We use the chain rule with $u(x) = -0{,}01x^2 + 0{,}8x - 12 = -0{,}01(x^2 - 80x + 1200) = -0{,}01(x-20)(x-60)$, $g(u) = \ln(u)$, $u'(x) = -0{,}02x + 0{,}8 = -0{,}02(x-40)$, $g'(u) = \frac{1}{u}$. Then $f'(x) = \frac{-0{,}02(x-40)}{-0{,}01(x-20)(x-60)} = \frac{2(x-40)}{(x-20)(x-60)}$. The equation $f'(x) = 0$ gives $2(x-40) = 0$ and hence $\underline{\underline{x = 40}}$ is the only stationary point for $f(x)$. By using a sign diagram we see that $f'(x)$ is positive for $20 < x < 40$ and negative for $40 < x < 60$. Hence $f(x)$ is strictly increasing in the interval $\langle 20, 40]$ and strictly decreasing in the interval $[40, 60\rangle$. Hence $\underline{\underline{x = 40}}$ is a local (and global) maximum point. To determine the curvature of the function we compute $f''(x)$. By using the quotient rule we get $f''(x) = \frac{-2(x^2-80+2000)}{(x-20)^2(x-60)^2}$. By completing the square we get $x^2 - 80 + 2000 = (x-40)^2 + 400$ which is greater or equal to 400 for all x. Hence we have that $f''(x)$ er negative in the whole interval $\langle 20, 60\rangle$ and hence $f(x)$ is concave in the entire domain of definition. Then we conclude that the stationary point $x = 40$ is a global maximum point and maximum is $\underline{\underline{f(40) = 2\ln(2)}}$.