

Key Problems

Problem 1.

Compute the definite integrals:

$$\begin{array}{lllll} \text{a) } \int_0^1 x \, dx & \text{b) } \int_0^1 x^2 \, dx & \text{c) } \int_0^1 x^3 \, dx & \text{d) } \int_0^1 e^x \, dx & \text{e) } \int_0^1 (e^x + e^{-x}) \, dx \\ \text{f) } \int_{-1}^1 x \, dx & \text{g) } \int_{-1}^1 x^2 \, dx & \text{h) } \int_{-1}^1 x^3 \, dx & \text{i) } \int_{-1}^1 e^x \, dx & \text{j) } \int_{-1}^1 (e^x + e^{-x}) \, dx \end{array}$$

Problem 2.

Compute the definite integrals:

$$\begin{array}{llll} \text{a) } \int_0^1 x e^x \, dx & \text{b) } \int_0^1 x \ln(x^2 + 1) \, dx & \text{c) } \int_0^1 \frac{1}{x^2 + 5x + 6} \, dx & \text{d) } \int_0^1 \frac{1}{x^2 + 4x + 4} \, dx \\ \text{e) } \int_{-1}^1 x e^x \, dx & \text{f) } \int_{-1}^1 x \ln(x^2 + 1) \, dx & \text{g) } \int_{-1}^1 \frac{1}{x^2 + 5x + 6} \, dx & \text{h) } \int_{-1}^1 \frac{1}{x^2 + 4x + 4} \, dx \end{array}$$

Problem 3.

Compute the definite integral $\int_1^2 1/x \, dx$. Then, for each value of n , estimate the area under the graph of $y = 1/x$ in the interval $[1,2]$ with a Riemann sum using n subintervals:

$$\text{a) } n = 2 \qquad \text{b) } n = 4 \qquad \text{c) } n = 8$$

Problem 4.

Compute the definite integral $\int_0^2 (x^3 - 3x + 1) \, dx$, and explain that the answer can be interpreted as $A_1 - A_2 + A_3$, where A_1, A_2, A_3 are the areas of three regions R_1, R_2, R_3 in the xy -plane bounded by the graph of $f(x) = x^3 - 3x + 1$ and the x -axis. Show these regions in a figure. A rough sketch is sufficient.

Problem 5.

Let R be the region bounded by the graph of $y = \ln(2 + x)$, the line $y = 2$, and the y -axis. Show the region R in a figure, and compute the area of R .

Problem 6.

Let R be the region bounded by the graphs of $y = x$ og $y = x^2$. Show the region R in a figure, and compute the area of R .

Problem 7.

Compute the integrals. Interpret each integral as an area, and show it in a figure.

$$\text{a) } \int_1^\infty 1/x \, dx \qquad \text{b) } \int_1^\infty 1/x^2 \, dx \qquad \text{c) } \int_0^1 -\ln x \, dx \qquad \text{d) } \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \qquad \text{e) } \int_0^\infty \frac{e^{1-\sqrt{x}}}{\sqrt{x}} \, dx$$

Problem 8.

Find the area under the graph of $y = 1/x$ in the interval $I = [1,2]$, and use this to show that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \right) = \ln(2)$$

Problem 9.

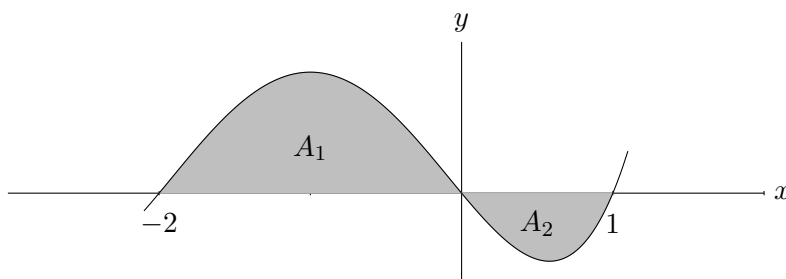
Exam problem MET11807 12/2019

Compute these integrals:

a) (6p) $\int 30x\sqrt{x} \, dx$

b) (6p) $\int xe^{-x} \, dx$

c) (6p) $\int \frac{6-3x}{4-9x^2} \, dx$



d) (6p) The graph of a function f is shown in the figure above. Determine the area A_1 when $A_2 = 22/15$ and

$$\int_{-2}^1 f(x) \, dx = \frac{18}{5}$$

Problem 10.

Optional: Problems from [Eriksen] (norwegian textbook)

Problem 5.6.1 - 5.6.5

Answers to Key Problems

Problem 1.

a) $1/2$

b) $1/3$

c) $1/4$

d) $e - 1$

e) $e - 1/e$

f) 0

g) $2/3$

h) 0

i) $e - 1/e$

j) $2(e - 1/e)$

Problem 2.

a) 1

b) $\ln(2) - 1/2$

c) $2\ln(3) - 3\ln(2)$

d) $1/6$

e) $2/e$

f) 0

g) $\ln(3) - \ln(2)$

h) $2/3$

Problem 3.

The definite integral $\int_1^2 1/x \, dx = \ln(2) \approx 0.693$, and the approximations are given by:

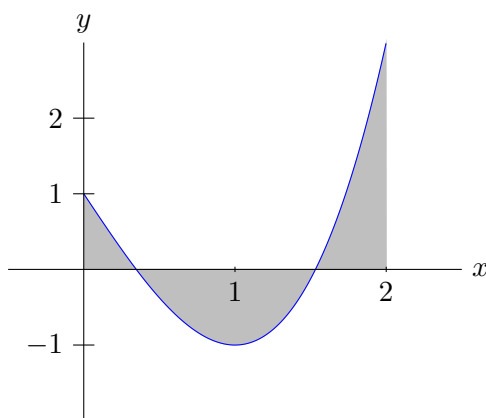
a) $0.5 \cdot 1/1 + 0.5 \cdot 2/3 \approx 0.833$

b) $0.25 \cdot 1/1 + 0.25 \cdot 4/5 + 0.25 \cdot 2/3 + 0.25 \cdot 4/7 \approx 0.760$

c) $0.125 \cdot 1 + 0.125 \cdot 8/9 + 0.125 \cdot 4/5 + 0.125 \cdot 8/11 + 0.125 \cdot 2/3 + 0.125 \cdot 8/13 + 0.125 \cdot 4/7 + 0.125 \cdot 8/15 \approx 0.725$

Problem 4.

The definite integral is $\int_0^2 (x^3 - 3x + 1) \, dx = 0$. The graph is shown below, and we see that the regions R_1 and R_3 are above the x -axis while the region R_2 is below. Hence $A_1 - A_2 + A_3 = 0$.

**Problem 5.**

$e^2 - 6 + \ln(4)$

Problem 6.

$1/6$

Problem 7.

a) ∞

b) 1

c) 1

d) $2e - 2$

e) $2e$

Problem 8.

The area is $\ln(2)$, and the Riemann sum for this area using n subintervals is

$$\frac{1}{n} \cdot \frac{1}{1} + \frac{1}{n} \cdot \frac{1}{1+1/n} + \cdots + \frac{1}{n} \cdot \frac{1}{1+(n-1)/n} = \frac{1}{n} + \frac{1}{n} \cdot \frac{n}{n+1} + \cdots + \frac{1}{n} \cdot \frac{n}{2n-1} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$$

The limit of this sum when $n \rightarrow \infty$ is therefore equal to the area $\ln(2)$.

Problem 9.

a) We write $x\sqrt{x} = x^{3/2}$ as a power, and use the power rule:

$$\int 30x\sqrt{x} \, dx = \int 30x^{3/2} \, dx = 30 \cdot \frac{2}{5} x^{5/2} + C = 12x^2\sqrt{x} + C$$

b) We use integration by parts with $u' = e^{-x}$ og $v = x$, which gives $u = -e^{-x}$ and $v' = 1$, and therefore

$$\int x e^{-x} dx = -e^{-x} \cdot x - \int -e^{-x} \cdot 1 dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C$$

c) We factorize the denominator as $4 - 9x^2 = (2 + 3x)(2 - 3x)$, and simplify the expression in the integral using partial fractions. This gives

$$\frac{6 - 3x}{4 - 9x^2} = \frac{A}{2 + 3x} + \frac{B}{2 - 3x} \Rightarrow 6 - 3x = A(2 - 3x) + B(2 + 3x)$$

Hence $6 - 3x = (2A + 2B) + (3B - 3A)x$, or $2A + 2B = 6$ and $3B - 3A = -3$. The system of equations can be written $A + B = 3$ and $B - A = -1$. This gives $2B = 2$ by adding the equations, which means that $B = 1$ and $A = 2$. The integral is therefore

$$\int \frac{6 - 3x}{4 - 9x^2} dx = \int \frac{2}{2 + 3x} + \frac{1}{2 - 3x} dx = \frac{2}{3} \ln |2 + 3x| - \frac{1}{3} \ln |2 - 3x| + C$$

d) We divide the integral in two parts and express it in terms of the areas A_1 and A_2 . This gives

$$\int_{-2}^1 f(x) dx = \int_{-2}^0 f(x) dx + \int_0^1 f(x) dx = A_1 - A_2$$

since $f(x) > 0$ in the interval $(-2,0)$ and $f(x) < 0$ in the interval $(0,1)$. We solve for A_1 , and find that

$$A_1 = \int_{-2}^1 f(x) dx + A_2 = \frac{18}{5} + \frac{22}{15} = \frac{76}{15} \approx 5.07$$