Key Problems

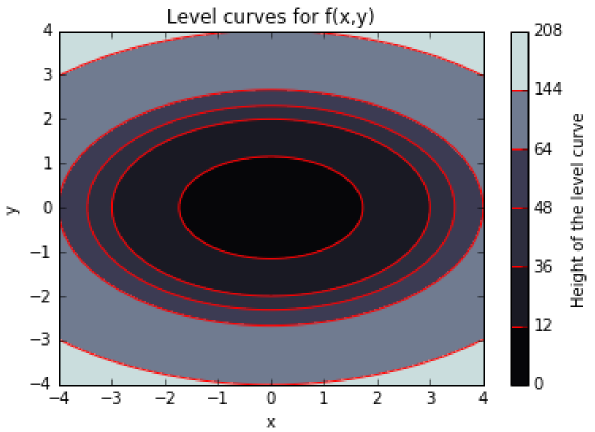
Problem 1.
We consider the region $D \subseteq \mathbb{R}^2$ given by the inequality $y(x-2) \leq 3$. Show $D = \{(x,y) : y(x-2) \leq 3\}$ in a figure, and mark the interior points and the boundary points of D. Is D compact?

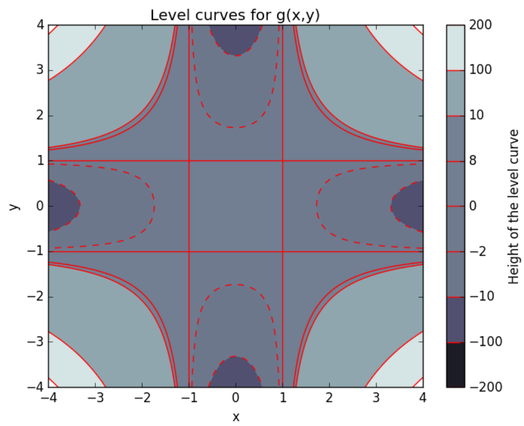
Problem 2.
We consider a subset D of the xy-plane $\mathbb{R}^2$ given by the following conditions. Determine if the subset is compact. It is useful to sketch the region D.

a) $2x + 3y = 6$
b) $2x + 3y < 6$
c) $2x + 3y \leq 6$
d) $x^2 + y^2 = 4$
e) $x^2 + y^2 \geq 4$
f) $x^2 + y^2 \leq 4$
g) $x^2 - 2x + 4y^2 = 4$
h) $x^2 - 2x + 4y^2 \leq 4$
i) $x^2 - 2x + 4y^2 \geq 4$
j) $xy = 1$
k) $xy \leq 1$
l) $xy \geq 1$
m) $\sqrt{x^2 + y^2} = 3$
n) $\sqrt{x^2 + y^2} \leq 3$
o) $x^2y^2 - x^2 - y^2 + 1 = 0$
p) $x^2y^2 - x^2 - y^2 + 1 = 1$

Problem 3.
What does the Extreme Value Theorem tell us? Given examples of a region D in the plane which is closed but not bounded, and a region E that is bounded but not closed. Can you find a function $f(x,y)$ that does not have a maximum and minimum in D, and a function $g(x,y)$ that does not have a maximum and minimum in E?

Problem 4.
The level curves of the functions $f(x,y) = 4x^2 + 9y^2$ and $g(x,y) = x^2y^2 - x^2 - y^2 + 1$ in the region $-4 \leq x,y \leq 4$ are shown in the figures below:





a) Find max / min $f(x,y)$ when $-4 \leq x,y \leq 4$ using the figure.
b) Find max / min $g(x,y)$ when $-4 \leq x,y \leq 4$ using the figure.
c) Find max / min $f(x,y)$ when $x^2 + y^2 = 16$ using the figure.
d) Find max / min $g(x,y)$ when $x = y$ using the figure.

Problem 5.
Solve the optimization problem:

a) $\max / \min f(x,y) = x^3 - 3xy + y^3$ when $0 \leq x,y \leq 1$

b) $\max / \min f(x,y) = x^3 - 3xy + y^3$ when $0 \leq x,y \leq 2$

c) $\max / \min f(x,y) = e^{xy-x-y}$ when $0 \leq x,y \leq 2$

d) $\max / \min f(x,y) = xy(x^2 - y^2)$ when $-1 \leq x,y \leq 1$

e) $\max / \min f(x,y) = (x^2 - 1)(y^2 - 1)$ when $-1 \leq x,y \leq 1$

Problem 6.
Problem 7.6.1 - 7.6.3 (norwegian textbook, optional)
Problem 9.27 - 9.31 (norwegian workbook, optional)

Answers to Key Problems

Problem 1.
Boundary points are given by the equation $y(x-2) = 3$, or points on the graph of $y = 3/(x-2)$ (a hyperbola). Interioer points are given by $y(x-2) < 3$, or points under the hyperbola when $x > 2$, and points over the hyperbola when $x < 2$, including all points with $x = 2$. The region D is not compact (it is closed but not bounded).

Problem 2.

a) No b) No c) No d) Yes e) No f) Yes g) Yes h) Yes

i) No j) No k) No l) No m) Yes n) Yes o) No p) No

Problem 4.

a) $f_{\text{min}} = 0$ at $(0,0)$, and $f_{\text{max}} = 208$ at $(\pm 4, \pm 4)$

b) $f_{\text{min}} = -15$ at $(0, \pm 4)$ and $(\pm 4, 0)$, and $f_{\text{max}} = 225$ at $(\pm 4, \pm 4)$

c) $f_{\text{min}} = 64$ at $(\pm 4, 0)$, and $f_{\text{max}} = 144$ at $(0, \pm 4)$

d) $f_{\text{min}} = 0$ at $(1,1)$ and $(-1,-1)$, and $f_{\text{max}} = 225$ at $(4,4)$ and $(-4,-4)$

Problem 5.

a) $f_{\text{max}} = 1$, $f_{\text{min}} = -1$

b) $f_{\text{max}} = 8$, $f_{\text{min}} = -1$

c) $f_{\text{max}} = 1$, $f_{\text{min}} = 1/e^2$

d) $f_{\text{max}} = 2\sqrt{3}/9$, $f_{\text{min}} = -2\sqrt{3}/9$

e) $f_{\text{max}} = 1$, $f_{\text{min}} = 0$