EBA2911 Mathematics for Business Analytics autumn 2019

Exercises

... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.

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Lecture 4

Sec. 10.4.4-10, 4.9.2, 6.11.4, 10.2, 10.3.2

Infinite geometric series and limits. Euler's number and continuous compounding of interest.

Here are recommended exercises from the textbook [SHSC].

Section **10.2** exercise 1-3, 5
Section **4.9** exercise 1-3, 9
Section **10.3** exercise 1b, 2b
Section **10.4** exercise 1-3
Section **10.5** exercise 6
Section **10.6** exercise 3

Problems for the exercise session Wednesday 4 Sept. from 14 o'clock in B2-085

Problem 1 Calculate the sum of the series.

(a) $1+1.04+1.04^2+1.04^3+\cdots+1.04^{10}$.
(b) $1+1.04+1.04^2+1.04^3+\cdots+1.04^{20}$.
(c) $1+1.04+1.04^2+1.04^3+\cdots+1.04^n$.
(d) $30\,000\cdot 1.04^{20}+30\,000\cdot 1.04^{19}+30\,000\cdot 1.04^{18}+\cdots+30\,000\cdot 1.04^2+30\,000\cdot 1.04$.
(e) Describe a financial situation where the sum in (d) is used.
(f) $1+\frac{1}{1.04}+\frac{1}{1.04^2}+\frac{1}{1.04^3}+\cdots+\frac{1}{1.04^{20}}$.
(g) Explain why 1.04^{20} multiplied with the sum in (f) gives the sum in (b).
(h) $1+\frac{1}{1.04}+\frac{1}{1.04^2}+\frac{1}{1.04^3}+\cdots+\frac{1}{1.04^n}$.
(i) $\frac{30\,000}{1.04}+\frac{30\,000}{1.04^2}+\frac{30\,000}{1.04^3}+\cdots+\frac{30\,000}{1.04^{20}}$.
(j) Describe a financial situation where the sum in (i) is used.

Problem 2 Suppose you are paid 500 000 every year for n years with the first payment in one year from now. Assume the interest is 3.5%.

(a) Write down the geometric series which gives the present value of the cash flow.
(b) Use the geometric series to compute the present value of the cash flow for $n=10$, $n=20$, $n=40$, $n=80$ and $n=1000$.
(c) Compute the present value of the cash flow if it continues forever.

Problem 3 The nominal annual interest is 4.8%.

(a) Assume annual compounding. Determine the annual growth factor. Determine the growth factor for 10 years. Determine the effective interest for 10 years.
(b) Assume quarterly compounding. Determine the annual growth factor and the effective interest. Determine the growth factor for 10 years. Determine the effective interest for 10 years.
(c) Assume monthly compounding. Determine the annual growth factor and the effective interest. Determine the growth factor for 10 years. Determine the effective interest for 10 years.
(d) Assume daily compounding. Determine the annual growth factor and the effective interest. Determine the growth factor for 10 years. Determine the effective interest for 10 years.

(e) Assume continuous compounding. Determine the annual growth factor and the effective interest. Determine the growth factor for 10 years. Determine the effective interest for 10 years.

Problem 4 You deposit 30 000 into an account with 2.9% nominal interest.

(a) Assume annual compounding.
  (i) Compute the balance after 10 years.
  (ii) Determine the growth factor and the relative change for the 10 years.

(b) Assume continuous compounding.
  (i) Compute the balance after 10 years.
  (ii) Determine the growth factor and the relative change for the 10 years.
  (iii) Determine the (annual) effective interest.

Problem 5 You consider an investment of 2 million in an asset which can be sold for 5 million after 20 years.

(a) With annual compounding, compute the internal rate of return.
(b) With quaterly compounding, compute the internal rate of return.
(c) With monthly compounding, compute the internal rate of return.
(d) With continuous compounding, compute the internal rate of return. (Hint: Try different rates.)

Problem 6

(a) Compute the present value of a payment of 30 million 5 years from now with 13% annual interest and continuous compounding.
(b) Compute the present value of the cash flow

Year	0	1	5	6	7
Payment	−70	−20	30	55	80

with 13% annual interest and continuous compounding.
(c) Suppose the cash flow describes an investment proposition. Determine if the investment is favorable.
(d) Show that the internal rate of return is about 10%. Describe what this means for the investment.
(e) Compute what the payment in year 0 should be for the internal rate of return to become 13% with continuous compounding and everything else as in (b).
(f) Compute the future value after 7 years for the cash flow in (b) and in (e).

Problem 7 Hege considers a mortgage with 25 annual payments. She believes she will be able to pay 120 000 each year. First payment is one year from now.

(a) Assume the interest is 2.0% with annual compounding. Determine the geometric series which gives the present value of the cash flow. Use it to calculate how much Hege can borrow.
(b) Assume the interest is 2.0% with continuous compounding. Determine the geometric series which gives the present value of the cash flow. Use it to calculate how much Hege can borrow.
(c) Compare the answers in (a) and (b).

Problem 8 We have an account with continuous compounding.

(a) Compute how much you have to deposit today for the balance to be 250 000 after 10 years if the interest is 3.4%.
(b) After 4 years the interest is changed to 1.9%. Calculate the balance after 10 years.
(c) Explain why the answer in (b) is given by the expression $\frac{250\,000}{e^{6(0.034-0.019)}}$.
(d) Calculate how much you would have to deposit in the case of (b) for the balance to become 250 000 after 10 years.
(e) Explain why the answer in (d) is given by the expression $\frac{250\,000}{e^{(4\cdot 0.034+6\cdot 0.019)}}$.
(f) Assume the interest for the first two years is 3.4%, the next three 1.9%, in year 6 and 7 it is 3.4% and the last three 1.9%. Compute how much you would have to deposit today to have 250 000 after 10 years.

Problem 9 Suppose you will be paid 300 000 every year in n years with the first payment a year from now. Suppose the interest is 3.5% with continuous compounding.

(a) Write down the geometric series which gives the present value of the cash flow.

(b) Use the geometric series to compute the present value of the cash flow for $n = 10$, $n = 20$, $n = 40$, $n = 80$ og $n = 1000$.
(c) Use the geometric series to compute the present value of the cash flow if it continues forever.

Problem 10 Suppose a constant amount $A = 40\,000$ (the annuity) is paid every year in n years with the first payment one year from now. Suppose the nominal interest is r with continuous compounding.
(a) Write down the geometric series which gives the present value of the cash flow if $n = 25$ og $r = 2.6\%$. Use this series to calculate the present value.
(b) Assume the annuity is paid forever. Write down the infinite geometric series which gives the present value of the cash flow if $r = 2.6\%$. Use this series to calculate the present value.
(c) Assume the annuity is paid forever. Determine the interest r such that the present value (K_0) becomes 3 million. (Hint: Try different rates.)
(d) Explain why (c) gives the equation

$$e^r = \frac{K_0 + A}{K_0} = \frac{3\,000\,000 + 40\,000}{3\,000\,000} = 1.0133$$

Answers

Problem 1
(a) $\frac{1.04^{11}-1}{0.04} = 13.49$.
(b) $\frac{1.04^{21}-1}{0.04} = 31.97$.
(c) $\frac{1.04^{n+1}-1}{0.04}$.
(d) $30\,000 \cdot 1.04 \cdot \frac{1.04^{20}-1}{0.04} = 929\,076.05$.
(e) A deposit of 30 000 every year for 20 years (starting today) into an account with 4% interest and annual compounding will give the sum as future value after 20 years.
(f) We read the geometric series backwards: $\frac{1}{1.04^{20}} \cdot \frac{1.04^{21}-1}{0.04} = 14.59$.
(g) $(1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \cdots + \frac{1}{1.04^{20}}) \cdot 1.04^{20} = 1.04^{20} + 1.04^{19} + \cdots + 1.04^2 + 1.04 + 1$.
(h) $\frac{1}{1.04^n} \cdot \frac{1.04^{n+1}-1}{0.04}$.
(i) $\frac{30\,000}{1.04^{20}} \cdot \frac{1.04^{20}-1}{0.04} = 407\,709.79$.
(j) The sum represents the present value (what you can borrow) for a 30 000 annuity (starting a year form now) with 4% interest and yearly compounding running for 20 years.

Problem 2
(a) $\frac{500\,000}{1.035} + \frac{500\,000}{1.035^2} + \frac{500\,000}{1.035^3} + \cdots + \frac{500\,000}{1.035^n}$.
(b) $n = 10 : \frac{500\,000}{1.035^{10}} \cdot \frac{1.035^{10}-1}{0.035} = 4\,158\,302.66$, $n = 20 : \frac{500\,000}{1.035^{20}} \cdot \frac{1.035^{20}-1}{0.035} = 7\,106\,201.65$,
$n = 40 : \frac{500\,000}{1.035^{40}} \cdot \frac{1.035^{40}-1}{0.035} = 10\,677\,536.17$, $n = 80 : \frac{500\,000}{1.035^{80}} \cdot \frac{1.035^{80}-1}{0.035} = 13\,374\,387.83$ and
$n = 1000 : \frac{500\,000}{1.035^{1000}} \cdot \frac{1.035^{1000}-1}{0.035} = 14\,285\,714.29$.
(c) $\frac{500\,000}{1.035^n} \cdot \frac{1.035^n-1}{0.035} = 500\,000 \cdot \frac{1-\frac{1}{1.035^n}}{0.035}$ which approaches $500\,000 \cdot \frac{1}{0.035} = 14\,285\,714.29$ more and more as n becomes bigger and bigger ("approches ∞").

Problem 3
(a) Annual growth factor: 1.048, growth factor for 10 years: $1.048^{10} = 1.5981$, effective interest for 10 years: 59.81%.
(b) Annual growth factor: 1.0489, growth factor for 10 years: 1.6115, effective interest for 10 years: 61.15%.
(c) Annual growth factor: 1.0491, growth factor for 10 years: 1.6145, effective interest for 10 years: 61.45%.
(d) Annual growth factor: 1.0492, growth factor for 10 years: 1.6160, effective interest for 10 years: 61.60%.
(e) Annual growth factor: 1.0492, growth factor for 10 years: 1.6161, effective interest for 10 years: 61.61%.

Problem 4

(a) (i) 39 927.77
(ii) Growth factor: 1.3309, relative change: 33.09%
(b) (i) 40 092.82
(ii) Growth factor: 1.3364, relative change: 33.64%
(iii) 2.94%

Problem 5

(a) $2.5^{\frac{1}{20}} - 1 = 4.69\%$
(b) 4.61%
(c) 4.59%
(d) Obtain the equation $e^r = 2.5^{\frac{1}{20}} = 1.0469$ and try: $r = 4.58\%$.

Problem 6

(a) 15.66 million.
(b) -14.49 million.
(c) One will not earn 13% interest on this investment.
(d) The present value 0.01 million is appoximately 0.
(e) 55.51 million.
(f) (b): -35.99 million, (e): 0.0 million.

Problem 7

(a) Present value:
$120\,000 \cdot \frac{1}{1.02} + 120\,000 \cdot \frac{1}{1.02^2} + 120\,000 \cdot \frac{1}{1.02^3} + \cdots + 120\,000 \cdot \frac{1}{1.02^{24}} + 120\,000 \cdot \frac{1}{1.02^{25}}$. Mortgage: 2 342 814.78
(b) Present value:
$120\,000 \cdot \frac{1}{e^{0.02}} + 120\,000 \cdot \frac{1}{(e^{0.02})^2} + 120\,000 \cdot \frac{1}{(e^{0.02})^3} + \cdots + 120\,000 \cdot \frac{1}{(e^{0.02})^{24}} + 120\,000 \cdot \frac{1}{(e^{0.02})^{25}}$.
Mortgage: $120\,000 \cdot \frac{1}{e^{0.02 \cdot 25}} \cdot \frac{e^{0.02 \cdot 25} - 1}{e^{0.02} - 1} = 2\,337\,286.57$

Problem 8

(a) 177 942.58
(b) 228 482.80
(d) 194 700.20
(f) 194 700.20

Problem 9

(a)
$$300\,000 \cdot \frac{1}{e^{0.035}} + 300\,000 \cdot \frac{1}{(e^{0.035})^2} + \cdots + 300\,000 \cdot \frac{1}{(e^{0.035})^{n-1}} + 300\,000 \cdot \frac{1}{(e^{0.035})^n}$$

(b) The sum of the geometric series: $300\,000 \cdot \frac{1}{(e^{0.035})^n} \cdot \frac{(e^{0.035})^n - 1}{e^{0.035} - 1}$. For $n = 10$: 2 487 206.55 for $n = 20$: 4 239 911.38 for $n = 40$: 6 345 389.07 for $n = 80$: 7 910 142.75 for $n = 1000$: 8 422 303.55.
(c) $300\,000 \cdot \frac{1}{(e^{0.035})^n} \cdot \frac{(e^{0.035})^n - 1}{e^{0.035} - 1} = 300\,000 \cdot \frac{1 - (e^{0.035})^{-n}}{e^{0.035} - 1}$ approaches $300\,000 \cdot \frac{1}{e^{0.035} - 1} = 8\,422\,303.55$ when n grows bigger and bigger.

Problem 10

(a)
$$\begin{aligned} &40\,000 \cdot \frac{1}{e^{0.026}} + 40\,000 \cdot \frac{1}{(e^{0.026})^2} + \cdots + 40\,000 \cdot \frac{1}{(e^{0.026})^{24}} + 40\,000 \cdot \frac{1}{(e^{0.026})^{25}} \\ &= 40\,000 \cdot \frac{1}{(e^{0.026})^{25}} \cdot \frac{(e^{0.026})^{25} - 1}{e^{0.026} - 1} = 40\,000 \cdot \frac{1}{e^{0.026 \cdot 25}} \cdot \frac{e^{0.026 \cdot 25} - 1}{e^{0.026} - 1} = 725\,796.53 \end{aligned}$$

(b)
$$\begin{aligned} &40\,000 \cdot \frac{1}{e^{0.026}} + 40\,000 \cdot \frac{1}{(e^{0.026})^2} + \cdots + 40\,000 \cdot \frac{1}{(e^{0.026})^n} + \ldots \\ &= 40\,000 \cdot \frac{1}{e^{0.026} - 1} = 1\,518\,548.20 \end{aligned}$$

(c) Obtain the equation $e^r = 1.0133$ and try: $r = 1.32\%$.