EBA2911, lecture 14, 4 Nov. 2020, Runar Ile

Plan

1. Implicit differentiation
2. The second derivative and curvature
3. Convex optimisation

1. Implicit differentiation

Ex $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = (-1) \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

– usual differentiation

Instead Put $y = f(x)$, so $y = \frac{1}{x} \quad | \cdot x$

get the equation $xy = 1$

Differentiate each side w.r.t. x and think of y as a function of x

$$(x \cdot y)'_x = (1)'_x$$

product rule gives

$$(x)'_x \cdot y + x \cdot (y)'_x = 0$$

$$1 \cdot y + x \cdot y' = 0$$

We can solve this equation for y'.

$$x \cdot y' = -y \qquad \Big| : x$$

$$y' = -\frac{y}{x}$$

(Note: $y = \frac{1}{x}$ so $y' = -\frac{\left(\frac{1}{x}\right)}{x} = -\frac{1}{x^2}$)

This is called implicit differentiation.

Can use this to find slopes of tangents to the curve.

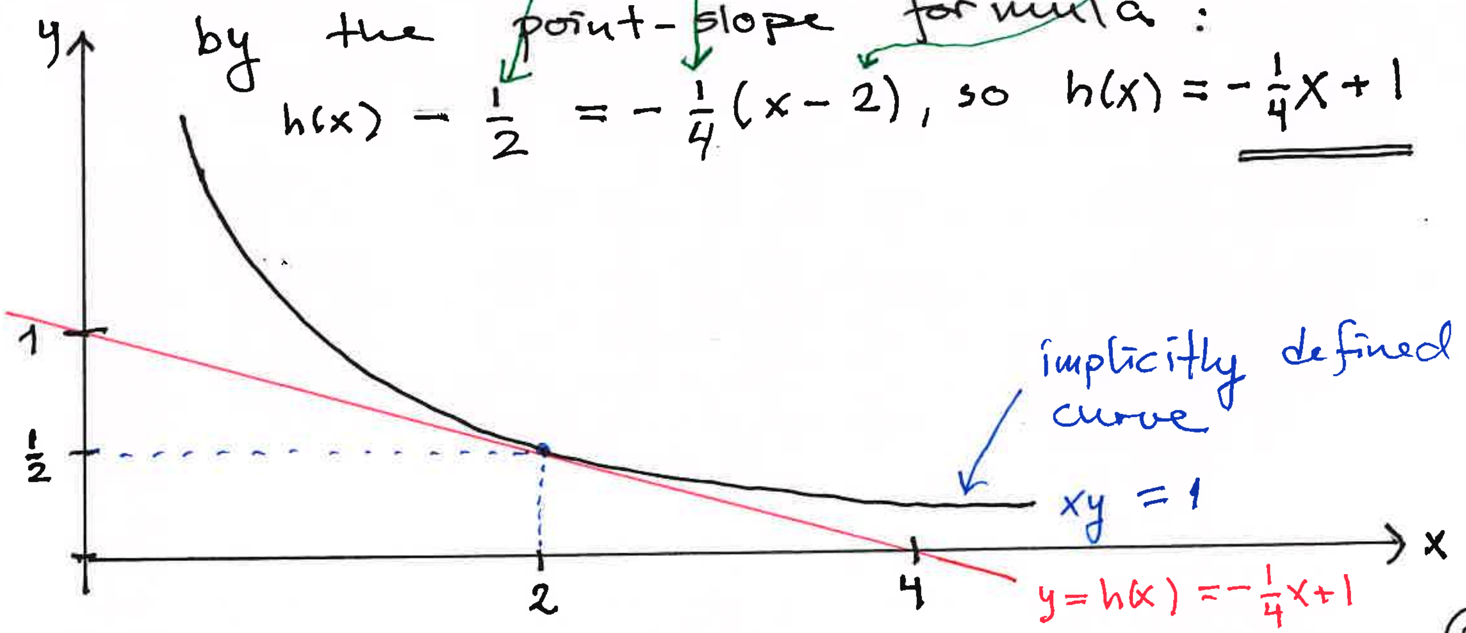
E.g. $\underline{x = 2}$ then $2 \cdot y = 1$ (the equation)

so $\underline{y = \frac{1}{2}}$

Also $y'\Big|_{\substack{x=2\\ y=\frac{1}{2}}} = -\frac{\left(\frac{1}{2}\right)}{2} = \underline{\underline{-\frac{1}{4}}}$

Can use this slope to find the function expression $h(x)$ for the tangent to the curve in the point $(2, \frac{1}{2})$ by the point-slope formula:

$h(x) - \frac{1}{2} = -\frac{1}{4}(x-2)$, so $\underline{\underline{h(x) = -\frac{1}{4}x + 1}}$



Ex A curve is implicitly defined by the equation $y^2 - x^3 = 1$

a) Express y' by x and y using implicit differentiation (think of y as a function of x)

b) Find all solutions for y when $x = 2$

c) Compute y' for these points.

Solution a) $(y^2)'_x - (x^3)'_x = (1)'_x$

Chain rule with: $u = y$ and $g(u) = u^2$

$u'_x = y'_x$ $g'(u) = 2u$

$= 2y$

$$2y \cdot y' - 3x^2 = 0$$

and solve for y'

$$2y \cdot y' = 3x^2$$

$$y' = \frac{3x^2}{2y}$$

b) $x = 2$. Solve $y^2 - 2^3 = 1$ for y.

$$y^2 = 9$$

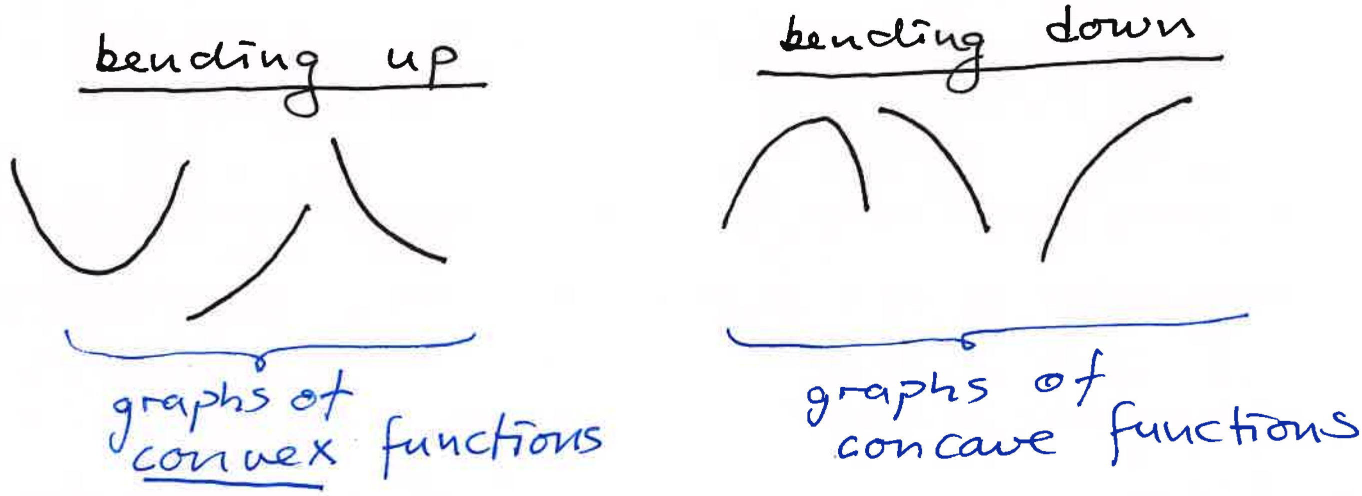
$$y = \pm 3$$

c) $(2, -3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot (-3)} = -2$

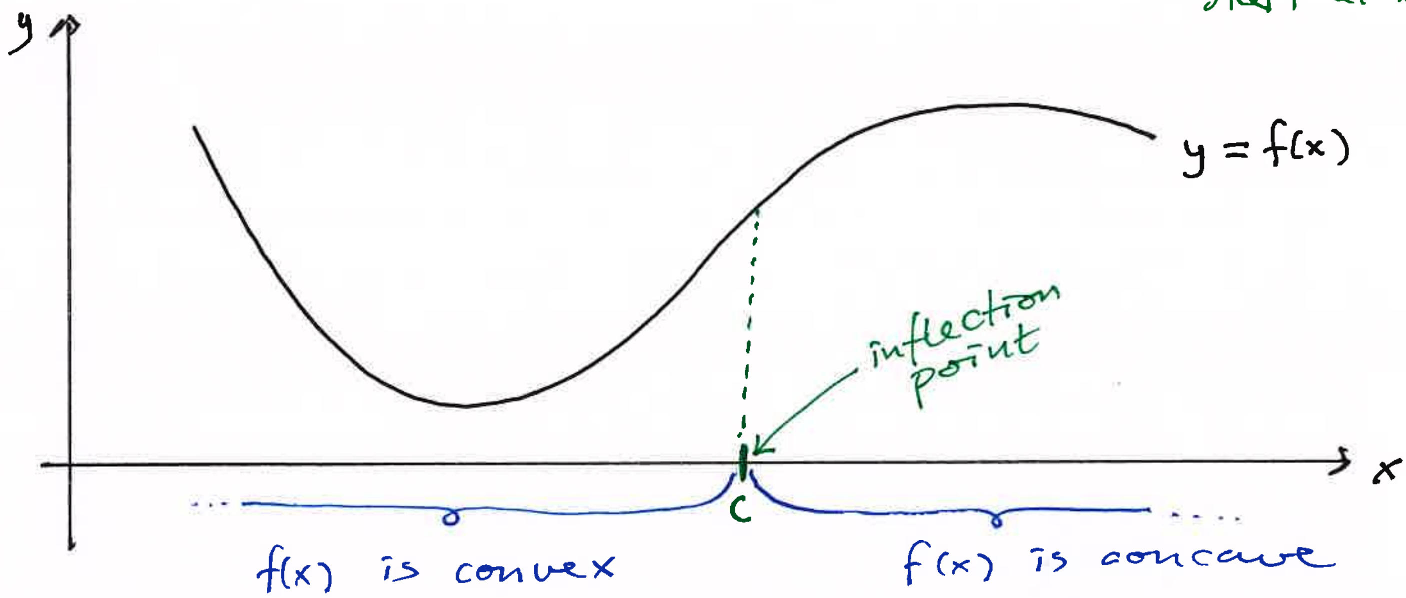
$(2, 3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot 3} = 2$

2. The second order derivative and curvature

In which direction does the graph bend?



Start at 10^{56}



Definition $f(x)$ is convex in the interval $[a, b]$ if $f''(x) \geq 0$ for all x in $\langle a, b \rangle$

(concave : $f''(x) \leq 0$ —— " ————)

A number c is an inflection point for $f(x)$ if $f''(x)$ changes sign at $x = c$.

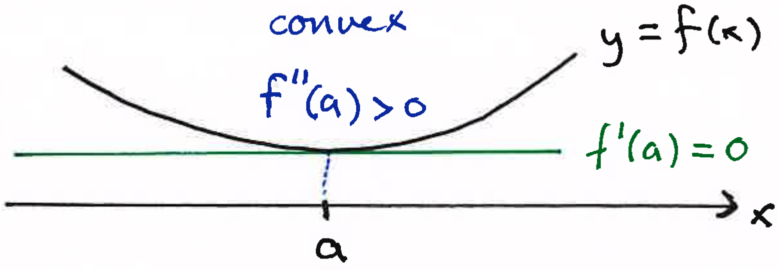
Note If $f(x)$ is convex, then $f'(x)$ is an increasing function.

If $f(x)$ is concave, —— " —— decreasing — " —

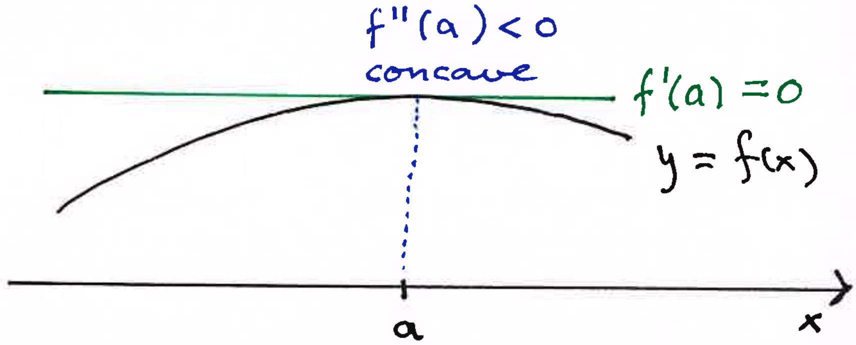
second derivative test (p 308)

Suppose $x = a$ is a stationary point for $f(x)$.

If $f''(a) > 0$ then $x = a$ is a (local) minimum point.



If $f''(a) < 0$ then $x = a$ is a (local) maximum point.



Ex $f(x) = x^3 - 3x^2 + 5$, local max/min?

$f'(x) = 3x^2 - 6x$

stationary points: solutions to $f'(x) = 0$

that is $3x^2 - 6x = 0$

take $3x$ outside a paranthesis

$3x(x - 2) = 0$

either $\underline{x = 0}$ or $\underline{x = 2}$

$f''(x) = [f'(x)]' = [3x^2 - 6x]' = 6x - 6$

Use the second derivative test:

$f''(0) = 6 \cdot 0 - 6 = -6 < 0$

so $x = 0$ is a (local) maximum point

$f''(2) = 6 \cdot 2 - 6 = 6 > 0$

so $x = 2$ is a (local) minimum point

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3. Convex optimisation

If $f(x)$ is convex everywhere in its domain, then any stationary point will be a global minimum point.

(and if $f(x)$ is concave everywhere: stationary points are global maximum points)

Ex $f(x) = x^4 + 5x^2 + 3$ with D_f = the whole number line

Find the stationary points of $f(x)$.

Determine if they are global maximum or minimum points.

Solution Calculate $f'(x) = 4x^3 + 10x$

Stationary points: solutions to eq $f'(x) = 0$

that is $4x^3 + 10x = 0$

$x(4x^2 + 10) = 0$ a product equal to 0

either $\underline{\underline{x = 0}}$ or $4x^2 + 10 = 0$

– only stationary point. – no solutions because $4x^2 + 10 \geq 10$ for all x

Calculate $f''(x) = 12x^2 + 10$

which is greater or equal to 10 for all x.

Hence $f(x)$ is convex on the whole number line. By convex optimisation the stationary point $x = 0$ is a global minimum point.

(and $f(0) = 3$ is the global minimum value of $f(x)$)