EBA 2911, lecture 15, 11 Nov. 2020, Runar Ile

Plan
1. l'Hôpitals rule
2. Marginal cost, average unit cost, marginal revenue

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1. l'Hôpital's rule

Limits of the type $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$

Notation $\lim_{x\to 5} f(x)$ is the number which $f(x)$ is approaching when x is approaching 5.

Ex $f(x) = \dfrac{3x-3}{\ln(x)}$. Want to find $\lim_{x\to 1} f(x)$.

$$\left.\begin{array}{ll}\text{Numerator: } & 3x-3 \xrightarrow[x\to 1]{} 3\cdot 1 - 3 = 0 \\ \text{Denominator: } & \ln(x) \xrightarrow[x\to 1]{} \ln(1) = 0\end{array}\right\} \frac{0}{0}\text{-expression}$$

Then we can use l'Hôpital's rule to proceed:

$$\lim_{x\to 1} f(x) \overset{\text{l'Hôp.}}{=} \lim_{x\to 1} \frac{(3x-3)'}{[\ln(x)]'} = \lim_{x\to 1} \frac{3}{\frac{1}{x}} = \frac{3}{\frac{1}{1}} = \underline{\underline{3}}$$

Note Has to be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$!

Ex Use l'Hôpital's rule to determine the limit

$$\lim_{x\to 0} \frac{x}{e^x - 1}$$

Solution Numerator: $x \xrightarrow[x\to 0]{} 0$

Denominator: $e^x - 1 \xrightarrow[x\to 0]{} e^0 - 1 = 1 - 1 = 0$

so we have a $\frac{0}{0}$-situation.

Then we can apply l'Hôpital's rule:

$(x)' = 1$ and $(e^x - 1)' = e^x$

$$\lim_{x\to 0} \frac{x}{e^x - 1} \overset{\text{l'Hôp.}}{=} \lim_{x\to 0} \frac{1}{e^x} = \frac{1}{e^0} = \frac{1}{1} = \underline{\underline{1}}$$

Ex $$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'Hôp.}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'Hôp.}}{=} \lim_{x\to\infty} \frac{2}{e^x} = \underline{\underline{0}}$$

$\frac{\infty}{\infty}$ $\qquad$ $\frac{\infty}{\infty}$

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2. Marginal cost, average unit cost, marginal revenue

$C(x)$ is the cost of producing x units of some commodity.

$C'(x)$ is the marginal cost.

Interpretation: The cost of producing one more unit than x units

$$= C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1} \approx \lim_{h\to 0} \frac{C(x+h) - C(x)}{h} = C'(x)$$

Why $C'(x)$? – much simpler math. to work with!

$R(x)$ is the <u>revenue</u> by selling x units

$R'(x)$ is the marginal revenue function

<u>Ex</u> x = tons of sold salmon

$R'(50)$ = extra revenue by selling 1 extra ton of salmon more than 50 tons.
(so $\approx R(51) - R(50)$).

<u>The profit function</u> (x = produced and sold units)

$P(x) = R(x) - C(x)$ (economists: $\pi(x)$)

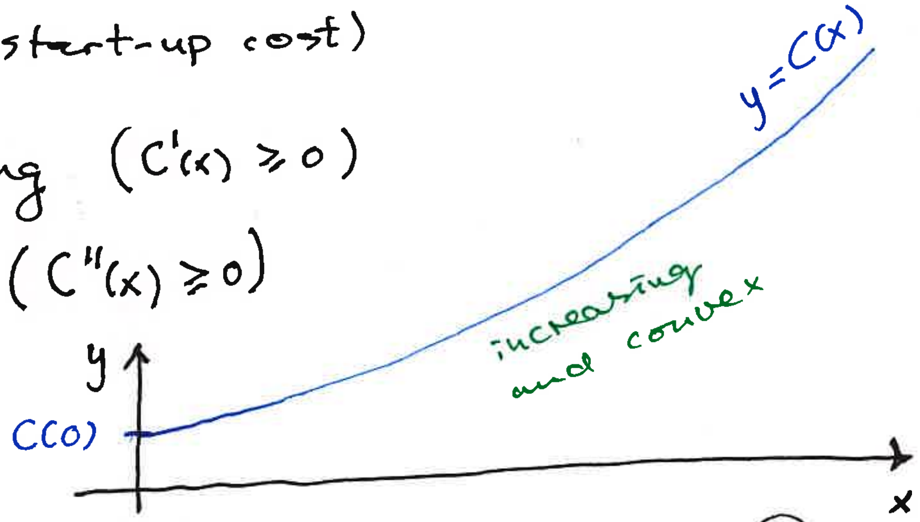
$P'(x)$ is the marginal profit function

<u>Average unit cost</u> of producing x units is $A(x) = \frac{C(x)}{x}$

"cost per unit" – not a constant function!

<u>Definition</u> $C(x)$ is a cost function if

1. $C(0) > 0$ (start-up cost)
2. $C(x)$ is increasing ($C'(x) \geq 0$)
3. $C(x)$ is convex ($C''(x) \geq 0$)



start at 11.00.

<u>Definition</u> If $x = c$ is a minimum point for $A(x)$ then c is called the cost optimum (the x-value that gives the minimal average unit cost)

<u>Result</u> If $C''(x) > 0$, then the cost optimum is the solution of the equation

$$C'(x) = A(x)$$

<u>Reason</u> We determine the stationary point of $A(x)$:

$$A'(x) = \left[\frac{C(x)}{x}\right]'$$

$$\overset{\text{quot. r.}}{=} \frac{C'(x)\cdot x - C(x)\cdot 1}{x^2} \quad \left| \frac{:x}{:x} \right.$$

$$= \frac{C'(x) - A(x)}{x}$$

So $A'(x) = 0$ is equivalent to $C'(x) = A(x)$

Assume $x = c$ is such a stationary point.
We use the second derivative test:
If $A''(c) > 0$, then c is a (loc.) min. point.

$$A''(x) \overset{\text{quot.r}}{=} \frac{[C'(x) - A(x)]' \cdot x - [C'(x) - A(x)] \cdot 1}{x^2}$$

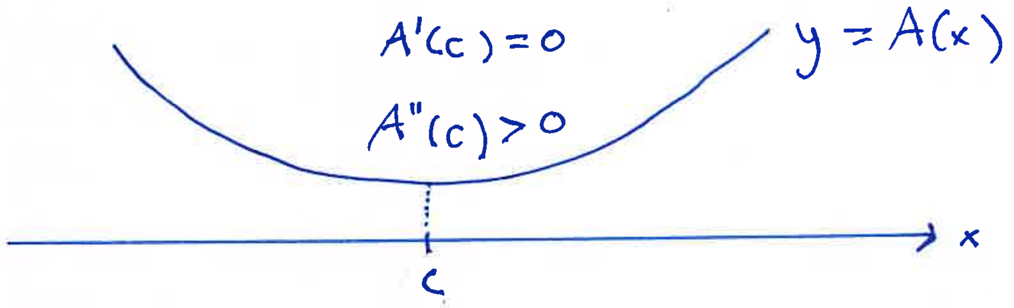
$$= \frac{[C''(x) - A'(x)] \cdot x - [C'(x) - A(x)]}{x^2}$$

Substitute $x = c$:

$$A''(c) = \frac{[C''(c) - \overbrace{A'(c)}^{=0}] \cdot c - [\overbrace{C'(c) - A(c)}^{=0}]}{c^2}$$

$$= \frac{C''(c) \cdot \cancel{c}}{c^{\cancel{2}}} = \frac{C''(c)}{c} > 0$$

(for $c > 0$)



<u>Ex</u> $C(x) = x^2 + 200x + 160\,000$

This is a cost function because :

① $C(0) = 160000 > 0$

② $C'(x) = 2x + 200 > 0$ for $x \geq 0$

③ $C''(x) = 2 > 0$ (for all x !)

The cost optimum is (by the result) the solution of the equation

$$C'(x) = A(x)$$

that is $$2x+200 = \frac{x^2+200x+160\,000}{x}$$

that is $$2x+200 = x+200+\frac{160\,000}{x}$$

so $$x = \frac{160\,000}{x}$$

that is $$x^2 = 160\,000$$

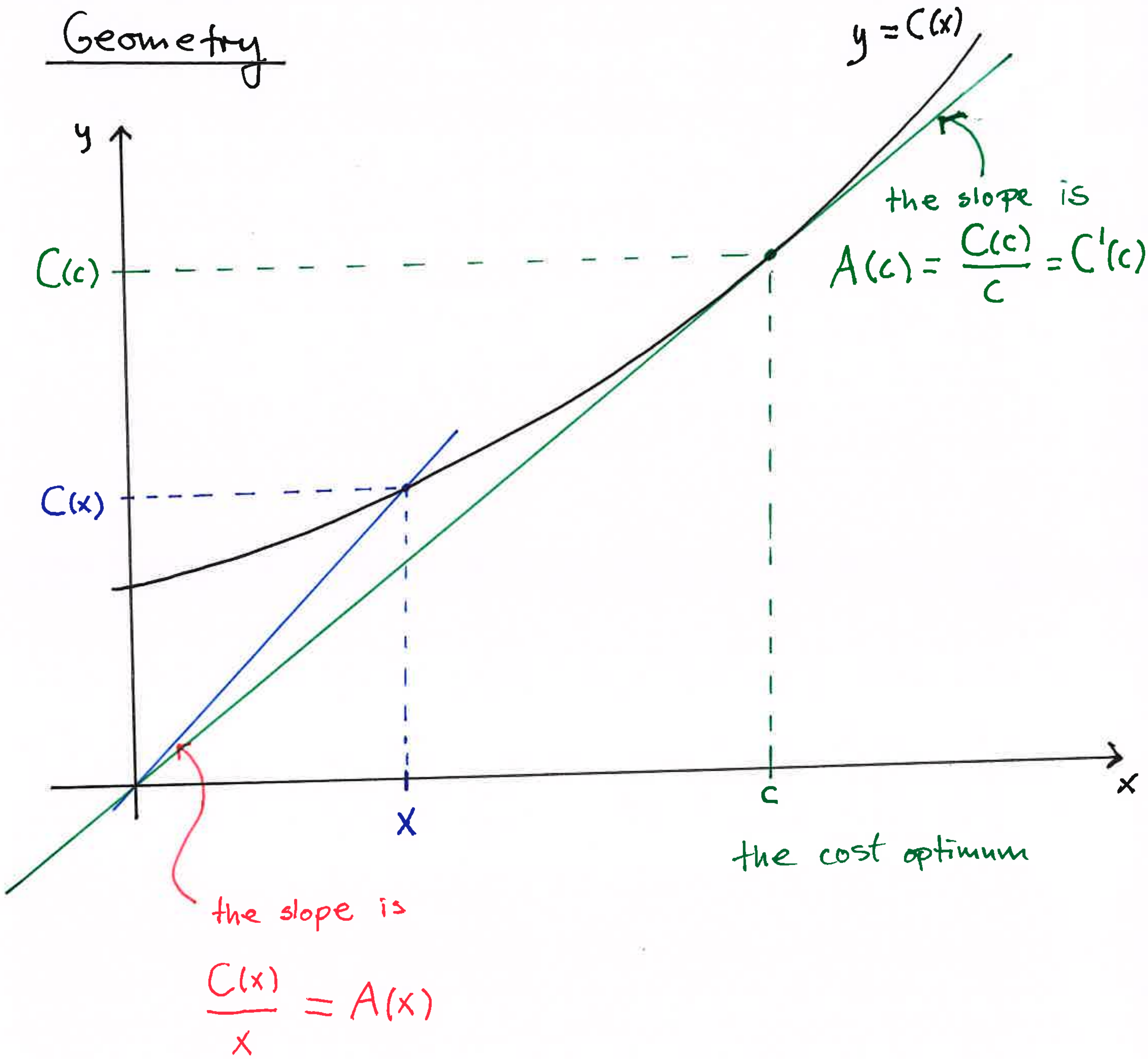
so $$\underline{\underline{x = 400}}$$ (only pos. x)

is the cost optimum.

The minimal average unit cost

is $A(400) = C'(400) = 2\cdot 400+200 = \underline{\underline{1000}}$

Geometry



and $A(c) = \frac{C(c)}{c}$ is the minimal average unit cost (the smallest slope!!)