EBA 2911, lecture 16, 18 Nov. 2020, Runar Ile

Plan 1. Elasticity

2. Linear approximation

3. Taylor polynomials

1. Elasticity

p = price/unit

$D(p)$ = demand of commodity with price p

= # sold units

The problem of comparing different units.

Ex: A barrel of North Sea crude oil costs \$ 43.95

A litre of — " — NOK 2.50

The price elasticity of the demand is

$$\varepsilon = \frac{\text{relative change in demand}}{\text{relative change in price}}$$

← numbers independent of choice of units

Ex In a month the price of a commodity drops from 12 thousand to 10 thousand, and the demand increases from 50 mill to 60 mill.

Then

$$\varepsilon = \frac{\left(\frac{60-50}{50}\right)}{\left(\frac{10-12}{12}\right)} = \frac{\left(\frac{10}{50}\right)}{\left(-\frac{2}{12}\right)} = \frac{120}{-100} = \underline{\underline{-1,2}}$$

Interpretation: if the price increases by 1% the demand decreases by 1.2%

Suppose the price is changed from p to $p+h$

Then the relative change in price is

$$\frac{p+h-p}{p} = \frac{h}{p}$$

$$\frac{\text{relative change in demand}}{\text{relative change in price}}$$

$$= \frac{\left(\dfrac{D(p+h) - D(p)}{D(p)}\right)}{\left(\dfrac{h}{p}\right)} \qquad \Big| \cdot \frac{p \cdot D(p)}{p \cdot D(p)}$$

$$= \frac{D(p+h) - D(p)}{h} \cdot \frac{p}{D(p)}$$

$$\Big\downarrow h \to 0 \qquad \text{(the change of price approaches 0)}$$

$$\varepsilon(p) = D'(p) \cdot \frac{p}{D(p)}$$

This is the momentary price elasticity of the demand function.

Is the revenue going up or down if we increase the price?

Revenue $R(p) = p \cdot D(p)$

Then the marginal revenue w.r.t. price is

$$R'(p) \overset{\text{product rule}}{=} 1 \cdot D(p) + p \cdot D'(p)$$

$$= D(p) \cdot \left[1 + \frac{p \cdot D'(p)}{D(p)} \right]$$

$$= \underbrace{D(p)}_{\text{always pos.}} \cdot \underbrace{\left[1 + \varepsilon(p) \right]}_{\text{positive or negative?}}$$

If $\varepsilon(p) < -1$

we get neg. $R'(p)$

so $R(p)$ is <u>decreasing</u>

- get <u>elastic</u> demand

If $\varepsilon(p) > -1$

we get positive $R'(p)$

so $R(p)$ is <u>increasing</u>

- get <u>inelastic</u> demand

If $\varepsilon(p) = -1$

the demand is unit elastic

<u>Ex</u> $D(p) = 50 - p$ for $0 < p < 50$

Then $D'(p) = -1$ and $\varepsilon(p) = \frac{D'(p) \cdot p}{D(p)} = \frac{(-1) \cdot p}{50 - p}$

$$= \frac{-p}{50-p}$$

Question: In what range do we have elastic demand?

Have to solve the inequality $\varepsilon(p) < -1$

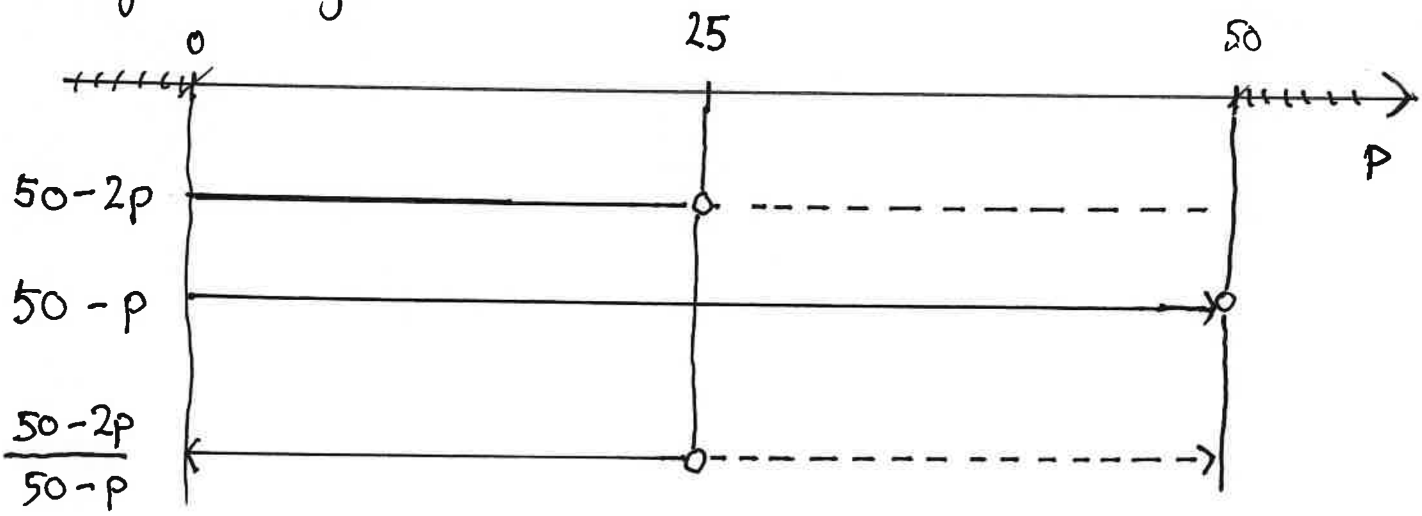
that is $\dfrac{-p}{50-p} < -1 \quad \big|\ +1$

so $-\dfrac{p}{50-p} + 1 < 0$

and $\dfrac{-p + (50-p)}{50-p} < 0$

get $\dfrac{50-2p}{50-p} < 0$

Sign diagram:



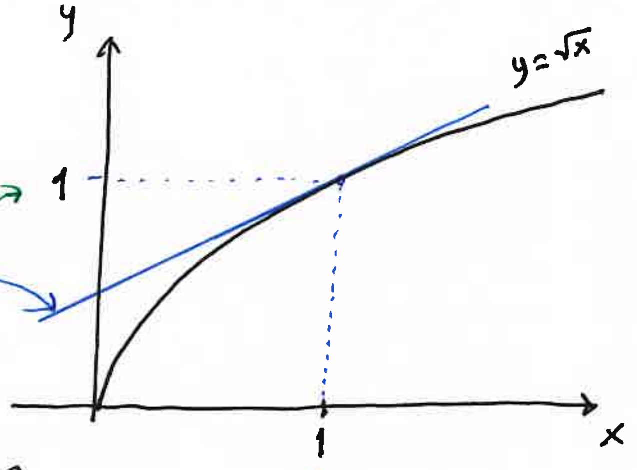
So elastic demand for p in $\langle 25, 50 \rangle$

and inelastic demand for p in $\langle 0, 25 \rangle$

and unit elastic demand for $p = 25$

Start again at 10.58

2. Linear approximation



Ex $f(x) = \sqrt{x}$

The linear approximation of $f(x)$ at $x = 1$.

We can find the expression for the tangent line by the point-slope formula

$$y - 1 = f'(1) \cdot (x - 1)$$

so $y = \underset{f(1)}{\underset{\shortparallel}{1}} + f'(1) \cdot (x-1)$

$$= \overset{\shortparallel}{1} + \tfrac{1}{2} \cdot (x-1)$$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} \cdot x^{\frac{1}{2}-1}$

$= \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}}$

$= \frac{1}{2\sqrt{x}}$

so $f'(1) = \frac{1}{2 \cdot \sqrt{1}}$

$= \frac{1}{2}$

This is a function in x denoted

$$P_1(x) = 1 + \tfrac{1}{2}(x-1)$$

It is called the degree 1 Taylor polynomial to $\sqrt{x}$ at 1.

Ex $P_1(1.1) = 1 + \frac{1}{2} \cdot (1.1 - 1) = 1.05$ – an approximation to $\sqrt{1.1}$.

check: $\sqrt{1.1} = 1.04881\ldots$

3. Taylor polynomials

<u>Ex</u> $f(x) = \sqrt{x}$

The Taylor polynomial of degree 2 to $\sqrt{x}$ at 1 is

$$P_2(x) = \underbrace{f(1) + f'(1)\cdot(x-1)}_{P_1(x)} + \frac{f''(1)}{2}\cdot(x-1)^2$$

$$= 1 + \tfrac{1}{2}(x-1) + \frac{\left(-\frac{1}{4}\right)}{2}\cdot(x-1)^2$$

$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2$$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}\cdot x^{-\frac{1}{2}}$$

$$f''(x) = \frac{1}{2}\left(x^{-\frac{1}{2}}\right)'$$

$$= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{1}{2}-1}$$

$$= -\frac{1}{4}\cdot x^{-\frac{3}{2}}$$

$$= -\frac{1}{4x\sqrt{x}}$$

so $f''(1) = -\dfrac{1}{4\cdot 1\cdot\sqrt{1}} = -\dfrac{1}{4}$

Pattern:

$$\boxed{P_2(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2}$$

- so $a = 1$ in the example.

$$\sqrt{2} = f(2) \approx P_2(2) = 1 + \tfrac{1}{2}(2-1) - \tfrac{1}{8}(2-1)^2$$

$$= 1 + \tfrac{1}{2} - \tfrac{1}{8} = 1.375$$

(check: $\sqrt{2} = 1.41421\ldots$) — an approximation to $\sqrt{2}$.

$$P_2(1.2) = 1 + \tfrac{1}{2}(1.2-1) - \tfrac{1}{8}\overbrace{(1.2-1)^2}^{0.2^2 = 0.04}$$

$$= 1 + 0.1 - 0.005 = 1.0950$$

(check: $\sqrt{1.2} = 1.0954$)

Ex $f(x) = \sqrt{x}$ in 1

Then the Taylor polynomial of degree 3 to $f(x)$ at 1 is:

$$P_3(x) = P_2(x) + \frac{f'''(1)}{6}\cdot(x-1)^3$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{\left(\frac{3}{8}\right)}{\cancel{6}\ 2}(x-1)^3$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

$$f''(x) = -\frac{1}{4x\sqrt{x}} = -\frac{1}{4}\cdot x^{-\frac{3}{2}}$$

$$f'''(x) = \left(-\frac{1}{4}\right)\cdot\left(-\frac{3}{2}\right)\cdot x^{-\frac{3}{2}-1} = \frac{3}{8}\cdot x^{-\frac{5}{2}} = \frac{3}{8x^2\sqrt{x}}$$

so $f'''(1) = \frac{3}{8\cdot 1^2\cdot\sqrt{1}} = \frac{3}{8}$

$$P_3(1.2) = 1 + \frac{1}{2}(1.2-1) - \frac{1}{8}(1.2-1)^2 + \frac{1}{16}(1.2-1)^3$$

$= 1.0955$ – a good approximation

(check: $\sqrt{1.2} = 1.0954$)

Pattern: $P_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2 + \frac{f'''(a)}{6}\cdot(x-a)^3$

The degree n Taylor polynomial for $f(x)$ in a is:

$$\boxed{P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n}$$

where $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$