Plan

1. Introduction to integration and definite integrals
2. Antiderivation and indefinite integrals
3. Integration rules

Teaching:

Fri 08-10 Lecture (new material)

Fri 10-13 Problem session

2 Fri 13-14 Lecture (review, go through selected problems)

Padlet

- during lecture
- in problem session
- problems to go through in the second part of the lecture (review)

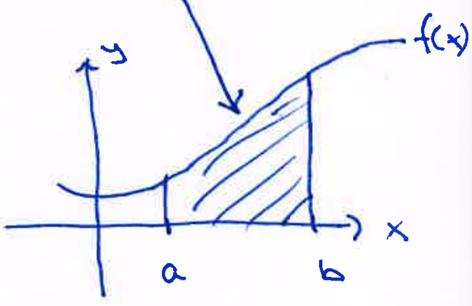
Topics:

- integration
- matrices and vectors
- functions in two variables

① Introduction to integration, definite integrals

Defn: Definite integral

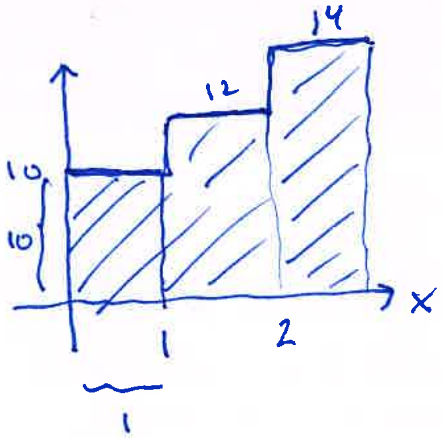
$$\int_a^b f(x)\,dx = \left\{ \begin{array}{l} \text{the area} \\ \text{under the graph of } f \\ \text{in } [a,b] \end{array} \right\}$$



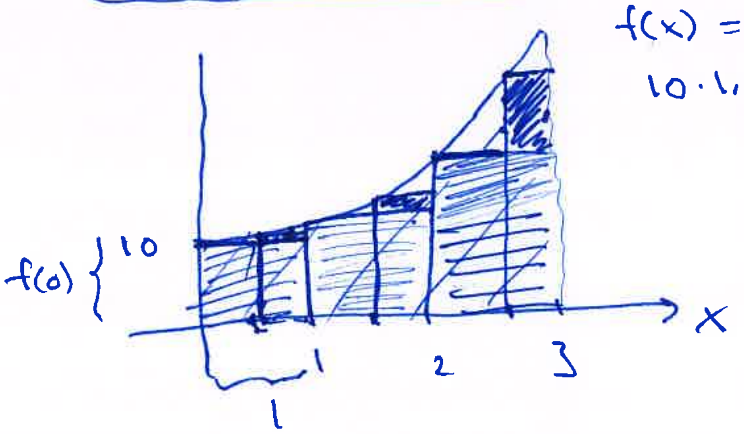
- f is cont. fn. on [a,b]
- $f(x) \geq 0$ in [a,b]
- $a < b$

Ex: Income

$10 + 12 + 14 = \underline{36}$



$f(x) = 10 \cdot 1{,}2^x$



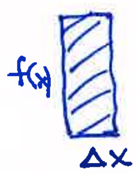
$$\text{Area} \simeq \underbrace{1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2)}_{\text{Riemann sum}}$$

$$\text{Area} \simeq \frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f(0.5) + \frac{1}{2} f(1) + \frac{1}{2} f(1.5) + \frac{1}{2} \cdot f(2) + \frac{1}{2} \cdot f(2.5)$$

Notation:

$$\int_a^b f(x)\,dx$$

- $\int$ = integration sign, sum
- $f(x)\,dx \rightarrow$



$$\frac{dy}{dx} = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

2 Antiderivatives and indefinite integrals

Defn: An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Ex: $f(x) = 2x$

Antiderivative: $F'(x) = 2x$

Answer: $F(x) = x^2$ is an antiderivative

$F(x) = x^2 + 1$,

$F(x) = x^2 + C$

are also antiderivatives

Defn: Indefinite integral

$$\int f(x)\,dx = \left\{\begin{matrix}\text{indefinite integral}\\ \text{of } f(x)\end{matrix}\right\} = \begin{matrix}\text{the general antiderivative}\\ \text{of } f(x)\end{matrix}$$

$= \text{all solutions of } F'(x) = f(x)$

Ex: $\int 2x\,dx = \underline{\underline{x^2 + C}}$ ← integration constant

Result: $\boxed{\text{If } f(x) \text{ has an antiderivative } F(x), \text{ then } \int f(x)\,dx = F(x) + C}$

$F'(x) = 2x$ and notice $F = x^2$ is one solution

$\Rightarrow x^2 + C$ are also solutions

$F'(x) = 2x$

$(x^2)' = 2x$

Subtract:

$F'(x) - (x^2)' = 2x - 2x$

$(F(x) - x^2)' = 0 \Rightarrow F(x) - x^2 = C \Rightarrow F(x) = \underline{x^2 + C}$

Ex: $\int (\underbrace{1+x^2}_{f(x)})\,dx = x + \frac{1}{3}\cdot x^3 + C$

$= x + \frac{x^3}{3} + C$

$(x)' = 1$

$(\frac{1}{3}x^3)' = \frac{1}{3}(x^3)' = \frac{1}{\cancel{3}}\cdot \cancel{3}x^2 = x^2$

dx : x is the integration variable

③ Integration rules

Derivation: $(x^n)' = nx^{n-1}$

Power rule: $\int x^n\,dx = ?$

$(x^{n+1})' = (n+1)x^n$

$\left(\frac{1}{n+1}x^{n+1}\right)' = \frac{1}{\cancel{n+1}}\cdot \cancel{(n+1)}x^n = x^n \quad (n \neq -1)$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ if $n \neq -1$

Ex: $\int x^2\,dx = \frac{1}{n+1}\cdot x^{n+1} + C = \frac{1}{3}x^3 + C$

$\int x^4\,dx = \frac{1}{5}x^5 + C$

$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-1}x^{-1} + C = -\frac{1}{x} + C$

$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{3/2}x^{3/2} + C = \frac{2}{3}x^{1+0.5} + C$

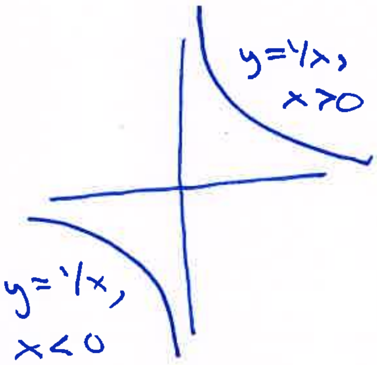
$n = 1/2$
$n+1 = 1/2 + 1 = 3/2$

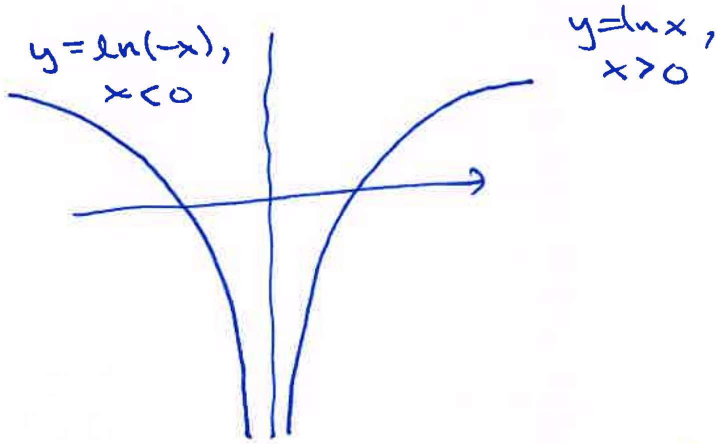
$= \frac{2}{3}x\cdot\sqrt{x} + C$

Ex: $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C$

$\frac{1}{x} = f(x)$

Explanation: Why $\int \frac{1}{x}\,dx = \ln|x| + C$





$f(x) = 1/x$, $x \neq 0$

$$F(x) = \left\{\begin{matrix} \ln x, & x>0 \\ \ln(-x), & x<0 \end{matrix}\right\} + C$$

$$= \ln|x| + C$$

$$(\ln(-x))' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

Integration rules:

(1) Power rule: $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $n \neq -1$

(2) $\int \frac{1}{x}\,dx = \ln|x| + C$

(3) $\int u(x) \pm v(x)\,dx = \int u(x)\,dx \pm \int v(x)\,dx$

(4) $\int c\cdot u(x)\,dx = c\cdot\int u(x)\,dx$ (c const.)

(5) Exponential function:
$$\begin{cases} \int e^x\,dx = e^x + C \\ \int a^x\,dx = \frac{1}{\ln(a)}\cdot a^x + C, \quad a>0 \end{cases}$$

Ex: $\int \underbrace{1 + 3x - x^2 + 2x^3} \, dx$

$= x + 3 \cdot \frac{x^2}{2} - \frac{x^3}{3} + 2 \cdot \frac{x^4}{4} + C$

$= \underline{x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{2}x^4 + C}$

$\int \frac{2 - x^2}{x} \, dx = \int \frac{2}{x} - \frac{x^2}{x} \, dx = \int 2 \cdot \frac{1}{x} - x \, dx$

$= \underline{2 \cdot \ln|x| - \frac{1}{2}x^2 + C}$

Integration techniques:

i) Substitution ← chain rule of derivation
ii) Integration by parts ← product rule — " —
iii) Partial fractions ← quotient rule — " —

Substitution:

Ex: $\int e^{-2x} \, dx = \cancel{\int e^{u} = e^{u} + C}$ $\quad u = -2x$

$\underline{(e^{-2x})' = e^{-2x} \cdot (-2)}$

$\int e^{-2x} dx = \int e^{u} \frac{du}{-2} = \int \underbrace{e^{u}\left(-\frac{1}{2}\right)} du = -\frac{1}{2} e^{u} + C$

$\boxed{\begin{array}{l} u = -2x \\ du = u' \cdot dx \end{array}}$

$du = -2 \cdot dx$

$dx = \frac{du}{-2}$

Why?
$u' = \frac{du}{dx}$
$u' dx = du$

$= \underline{\underline{-\frac{1}{2} e^{-2x} + C}}$

Ex: $\int x\sqrt{x^2+3}\,dx = \int x\sqrt{u}\cdot\frac{du}{2x}$

$$\boxed{\begin{aligned} u &= x^2+3 \\ du &= \underset{\uparrow\; u'}{2x}\,dx \end{aligned}}$$

$dx = \frac{du}{2x}$

$$\int \frac{\cancel{x}\sqrt{u}}{2\cancel{x}}\,du = \int \frac{1}{2}\underset{\underset{u^{1/2}}{\uparrow}}{\sqrt{u}}\,du = \frac{1}{\cancel{2}}\cdot\frac{1}{3/\cancel{2}}u^{3/2}+C$$

$n = 1/2$

$n+1 = 3/2$

$$= \frac{1}{3}u\sqrt{u}+C = \underline{\underline{\frac{1}{3}(x^2+3)\sqrt{x^2+3}+C}}$$

Substitution:

- choose $u = \ldots$ (expression in x) ← change of variables
- use change of variables and $du = u'\cdot dx$ to rewrite the integral of the form $\int \underset{\underset{\text{expr. in } u}{\uparrow}}{\_\_\_}\,du$
- solve the integral in u (hopefully easier than the original one) and express the answer in x.

Lecture 17, Part 2:

Review:

$$\int_a^b f(x)\,dx = \left\{\begin{matrix}\text{area under the graph} \\ \text{of } f \text{ in } [a,b]\end{matrix}\right\}$$ definite integral

$$\int f(x)\,dx = \left\{\begin{matrix}\text{the general} \\ \text{antiderivative of } f(x)\end{matrix}\right\}$$ indefinite integral

Computing indefinite integrals.

i) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad , \; n \neq -1$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

iii) $\int u \pm v\,dx = \int u\,dx \pm \int v\,dx$

iv) $\int c \cdot u\,dx = c \cdot \int u\,dx$

u, v : expressions in x
c : constant

v) $\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C \quad (a > 0)$

Substitution:

$\int f(x)\,dx =$

$\int g(u)\,du$

Method:

i) Choose a change of variables $u = \_\_\_$ (expr. in x)

ii) Use change of variables and $du = u' \cdot dx$ to transform the integral into one in u

iii) Solve $\int g(u)\,du$ and write it as an expression in x using change of variables.

Problems

Problemset 17

3a Find a fn. $f(x)$ such that $f'(x)=2$, $D_f=(\leftarrow,\rightarrow)$

$\int 2\,dx = 2x+C$ but $f(x)=2x$

$f(x)=2x+1$

4d $\int f(x)\,dx = xe^{2x}+C$

$\Rightarrow f(x) = (xe^{2x}+C)' = 1\cdot e^{2x} + x\cdot e^{2x}\cdot 2$

$= e^{2x}(1+2x) = (2x+1)e^{2x}$

h $\int xe^{2x}\,dx \neq \int x\,dx \cdot \int e^{2x}\,dx \leftarrow$ can use integration by parts

$\int xe^{2x}\,dx = \frac{1}{2}\left(xe^{2x} - \frac{1}{2}e^{2x}\right)+C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}+C$

$(xe^{2x})' = 2xe^{2x} + e^{2x}$

$(e^{2x})' = e^{2x}\cdot 2 = 2e^{2x}$

$(xe^{2x} - \frac{1}{2}e^{2x})' = 2xe^{2x} + \cancel{e^{2x}} - \frac{1}{\cancel{2}}\cdot\cancel{2}\cancel{e^{2x}}$

5. $\int \frac{(A+Bx)e^{2x}}{2\sqrt{x}}\,dx = \sqrt{x}\,e^{2x}+C$

$(\sqrt{x}\,e^{2x}+C)' = \frac{1}{2\sqrt{x}}\cdot e^{2x} + \sqrt{x}\,e^{2x}\cdot 2\cdot\frac{2\sqrt{x}}{2\sqrt{x}}$

$= \frac{e^{2x}+4x\,e^{2x}}{2\sqrt{x}} = \frac{(1+4x)e^{2x}}{2\sqrt{x}}$

$A=1$

$B=4$

8c: $\int e^{1-2x}\,dx = \int e^u \frac{du}{-2} = \int -\frac{1}{2} e^u \, du$

$\boxed{\begin{array}{l} u = 1-2x \\ du = -2\,dx \end{array}}$ $dx = \frac{du}{-2}$

$du = u'\,dx$

$= -\frac{1}{2} e^u + C = -\frac{1}{2} e^{1-2x} + C$

9d $\int \frac{x}{1+x^2}\,dx = \int \frac{x}{u} \frac{du}{2x} = \int \frac{\cancel{x}}{u \cdot 2\cancel{x}}\,du$

$\boxed{\begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \end{array}}$

$dx = \frac{du}{2x}$

$= \int \frac{1}{2} \cdot \frac{1}{u}\,du = \frac{1}{2} \cdot \ln|u| + C = \frac{1}{2} \ln(1+x^2) + C$

10. $\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\sqrt{x}} \cdot (-2\sqrt{x})\,du$

$1-u$

$\boxed{\begin{array}{l} u = 1-\sqrt{x} \\ du = -\frac{1}{2\sqrt{x}}\,dx \end{array}}$

$dx = -2\sqrt{x}\,du$

$= \int \frac{-2\cancel{\sqrt{x}}\,e^u}{\cancel{\sqrt{x}}}\,du = \int -2e^u\,du = -2e^u + C$

$= -2e^{1-\sqrt{x}} + C$

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^{1-u}}{u} \cdot 2\sqrt{x}\,du$$

$$\boxed{\begin{array}{l} u=\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$$

$$dx = 2\sqrt{x}\,du$$

$$\boxed{\begin{array}{l} v=1-u \\ dv = -1\cdot du \end{array}}$$

$$= \int \frac{2\cancel{\sqrt{x}}\,e^{1-u}}{\cancel{u}}\,du = \int 2e^{1-u}\,du = \int 2e^{v}\,\frac{dv}{-1}$$

$$= 2\,\frac{e^{1-u}}{-1} + C = -2e^{1-u} + C$$

$$= \underline{-2e^{1-\sqrt{x}} + C}$$

$\underline{8d}$. $\int 3^x\,dx = \underline{\underline{\frac{1}{\ln(3)}\cdot 3^x + C}}$

$a=3$