Plan

1 Substitution
2 Integration by parts
3 Integrals of rational functions

① Substitution.

$$\int f(x)\,dx = \int g(u)\,du$$

Method:

① Choose a change of variables $u = u(x)$ ← some expression in x

② Use $\boxed{\begin{array}{l} u = u(x) \\ du = u' \cdot dx \end{array}}$ to change the integral into an integral in u

③ Solve the integral in u, and express the answer in x.

Ex from Problem Set 17.

9c $\int x e^{-x^2} dx =$

$$\boxed{\begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array}} \qquad dx = \frac{du}{-2x}$$

$$= \int x e^u \cdot \frac{du}{-2x}$$

$$= \int \frac{\cancel{x} e^u}{-2\cancel{x}}\,du = \int -\frac{1}{2} e^u\,du$$

$$= -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = \underline{\underline{-\frac{1}{2}e^{-x^2} + C}}$$

9e

$$\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}} \cdot \cancel{x}\,du$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$dx = \frac{du}{1/x}\,\frac{\cdot x}{\cdot x} = \underline{x \cdot du}$$

$$= \int u\,du = \frac{1}{2}u^2 + C = \underline{\underline{\frac{1}{2}(\ln x)^2 + C}}$$

Ex:

$$\int \underline{e^{\sqrt{x}}}\,dx = \int e^u \cdot 2\sqrt{x}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$dx = 2\sqrt{x}\,du$$

$$= \int e^u\, 2u\,du = 2\underbrace{\int u e^u\,du}_{\text{integration by parts}}$$

$$= 2\left(ue^u - e^u + c\right) = \underline{2ue^u - 2e^u + C}$$

$$= \underline{\underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}}$$

(2) Integration by parts

"Product rule inverted"

Ex: $\int x \cdot \ln(x)\, dx$ ← integral of a product

$\cancel{\neq \int x\,dx \cdot \int \ln(x)\, dx}$

Product rule:

$(u \cdot v)' = u'v + u \cdot v'$

$\downarrow \int \ldots dx$

$uv = \int u'v\,dx + \int uv'\,dx$

$$\boxed{\int u' \cdot v\,dx = uv - \int uv'\,dx}$$

Formula for integration by parts.

Ex: $\int \underset{u'}{x} \cdot \underset{v}{\ln x}\, dx = \int u'v\,dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$= uv - \int uv'\,dx = \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{1}{2}x^2 + C$

$= \underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

Integration by parts:

$$\int u'v\,dx = uv - \int uv'\,dx$$

Ex:

$$\int \underbrace{x}_{u'} \cdot \underbrace{e^x}_{v}\,dx = \int u'v\,dx$$

$$\boxed{\begin{matrix} u = \frac{1}{2}x^2 & v = e^x \\ u' = x & v' = e^x \end{matrix}}$$

$$\cancel{= \frac{1}{2}x^2 \cdot e^x - \int \frac{1}{2}x^2 \cdot e^x\,dx}$$

$$\int \underbrace{x}_{v}\,\underbrace{e^x}_{u'}\,dx = \int u'v\,dx$$

$$\boxed{\begin{matrix} u = e^x & v = x \\ u' = e^x & v' = 1 \end{matrix}}$$

$$= e^x \cdot x - \int e^x \cdot 1\,dx = xe^x - \int e^x\,dx$$

$$\int xe^x\,dx = \underline{\underline{xe^x - e^x + C}}$$

Test: $(xe^x - e^x)' = \cancel{1 \cdot e^x} + x \cdot e^x - \cancel{e^x} = \underline{xe^x}$

Ex: $\int \ln(x)\,dx = \int \underset{u'}{1} \cdot \underset{\checkmark}{\ln(x)}\,dx$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= u \cdot v - \int x \cdot \frac{1}{x}\,dx \quad = uv - \int uv'\,dx$$

$$= x\ln(x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln(x) - \int 1\,dx = \underline{\underline{x\ln(x) - x + C}}$$

Integration rule: $\boxed{\int \ln x\,dx = x\ln(x) - x + C}$

③ Integrals of rational functions / fractions

Ex: $\int \frac{1}{1-x}\,dx$, $\int \frac{x}{1-x}\,dx$, $\int \frac{2}{1-x^2}\,dx$

Integration of rational fn:

Look at the degree of the denominator

Ex: $\int \frac{1}{1-x}\,dx$

(degree one)

$$\boxed{\begin{array}{l} u = 1-x \\ du = -1 \cdot dx \end{array}}$$

$$= \int \frac{1}{u}\frac{du}{-1} = \int -\frac{1}{u}\,du = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = \underline{\underline{-\ln|1-x| + C}}$$

Formula: $\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$, A, a, b constants, $a \neq 0$

Ex: $\int \frac{x}{1-x}\,dx = \int -1 + \frac{1}{1-x}\,dx$

Do polynomial division to simplify

$x : (-x+1) = -1$
$-(x-1)$
1

$\frac{x}{1-x} = -1 + \frac{1}{1-x}$

$= -x + \int \frac{1}{1-x}\,dx = -x - \ln|1-x| + C$

Ex: $\int \frac{x^2}{1-x}\,dx = \int -x-1 + \frac{1}{1-x}\,dx = -\frac{1}{2}x^2 - x - \ln|1-x| + C$

$x^2 : (-x+1) = -x-1$
$-(x^2-x)$
x
$-(x-1)$
1

$\frac{x^2}{1-x} = -x-1 + \frac{1}{1-x}$

Partial fractions:

Ex: $\int \frac{2}{1-x^2}\,dx = \int \frac{2}{u} \frac{du}{-2x} = \int -\frac{1}{xu}\,du$

$\boxed{\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array}}$

Partial fraction: $\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1-x} + \frac{1}{1+x}\,dx = -\ln|1-x| + \ln|1+x| + C$

$\frac{2}{1-x^2} = \frac{A^{=1}}{1-x} + \frac{B^{=1}}{1+x} \quad \Big| \cdot (1-x)(1+x)$

$2 = \frac{A}{\cancel{1-x}}\cancel{(1-x)}(1+x) + \frac{B}{\cancel{1+x}}(1-x)\cancel{(1+x)}$

$\underbrace{2}_{f(x)} = \underbrace{A\cdot(1+x) + B(1-x)}_{g(x)}$

Alt I: $2 = A + Ax + B - Bx$

$= (A+B) + (Ax - Bx)$

$2 = (A+B) + (A-B)x$

$\boxed{\begin{array}{l} A - B = 0 \\ A + B = 2 \end{array}}$

$B = A$

$A + A = 2$

$2A = 2$

$A = 1$

$B = 1$

Method:

① Factorize the denominator

$1-x^2 = (1-x)(1+x)$

② Rewrite:

$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$

③ Find constants A, B such that this holds.

Multiply with the common denom.

$1-x^2 = (1-x)(1+x)$

Alt 2:

$x = 1: \quad 2 = A\cdot 2 + B\cdot 0$

$x = -1: \quad 2 = A\cdot 0 + B\cdot 2$

$A = 1$

$B = 1$

Problem Set 18.

5g. $\int \frac{x^2}{(1-x)^2}\,dx = \int \frac{x^2}{1-2x+x^2}\,dx$

$$x^2 : (x^2-2x+1) = \underline{1}$$
$$\underline{-(x^2-2x+1)}$$
$$2x-1$$

$$\frac{x^2}{(1-x)^2} = 1 + \frac{2x-1}{x^2-2x+1}$$

$$= \int 1 + \frac{2x-1}{x^2-2x+1}\,dx = x + \int \frac{2x-1}{x^2-2x+1}\,dx$$

$$\frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \quad \Big| \cdot (x-1)^2$$

$$2x-1 = A\cdot(x-1) + B$$
$$2x-1 = Ax + (-A+B)$$

$A=2$ $\qquad$ $A=\underline{2}$

$-A+B=-1$ $\qquad$ $B=-1+2=\underline{1}$

$$\boxed{\frac{2x-1}{(x-1)^2} = \frac{2}{x-1} + \frac{1}{(x-1)^2}}$$

$$= x + \int \frac{2}{(x-1)} + \frac{1}{(x-1)^2}\,dx = x + \int \left(\frac{2}{u} + \frac{1}{u^2}\right) du$$

$$\boxed{\begin{aligned} u &= x-1 \\ du &= 1\cdot dx \end{aligned}}$$

$$= x + 2\ln|u| + \int u^{-2}\,du = x + 2\ln|u| + \frac{u^{-1}}{-1} + C$$

$$= x + 2\ln|x-1| - \frac{1}{x-1} + C$$

6b. $\int x \cdot \ln(1-x)\,dx =$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln(1-x) \\ u' = x & v' = \frac{1}{1-x}\cdot(-1) \end{array}}$$

$$= \frac{1}{2}x^2 \cdot \ln(1-x) - \int \frac{1}{2}x^2 \cdot (-1) \cdot \frac{1}{1-x}\,dx$$

$$= \frac{1}{2}x^2 \ln(1-x) + \frac{1}{2}\boxed{\int \frac{x^2}{1-x}\,dx}$$

$x^2 : (-x+1) = -x-1$

$-(x^2 - x)$

x

$-(x-1)$

1

$$\frac{x^2}{1-x} = -x - 1 + \frac{1}{1-x}$$

$$= \frac{1}{2}x^2 \ln(1-x) + \frac{1}{2}\int -x - 1 + \frac{1}{1-x}\,dx$$

$$= \frac{1}{2}x^2 \ln(1-x) + \frac{1}{2}\left(-\frac{1}{2}x^2 - x - \ln|1-x|\right) + C$$

$$= \frac{1}{2}x^2 \ln(1-x) - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2}\ln|1-x| + C$$

9.
$$\int \frac{\sqrt{x}+1}{1-\sqrt{x}}\,dx = \int \frac{\sqrt{x}+1}{u}(-2\sqrt{x})\,du$$

$$\boxed{\begin{array}{l} u = 1-\sqrt{x} \\ du = -\frac{1}{2\sqrt{x}}\,dx \end{array}} \rightarrow \sqrt{x} = 1-u$$

$$dx = -2\sqrt{x}\,du$$

$$= \int \frac{-2\sqrt{x}(\sqrt{x}+1)}{u}\,du = \int \frac{-2(1-u)(2-u)}{u}\,du$$

$$= \int \frac{-2(2-3u+u^2)}{u}\,du = \int -2u+6-\frac{4}{u}\,du$$

$$= -u^2+6u-4\ln|u|+C$$

$$= \underline{\underline{-(1-\sqrt{x})^2+6(1-\sqrt{x})-4\ln|1-\sqrt{x}|+C}}$$

$$= 5-4\sqrt{x}-x-4\ln|1-\sqrt{x}|+C$$

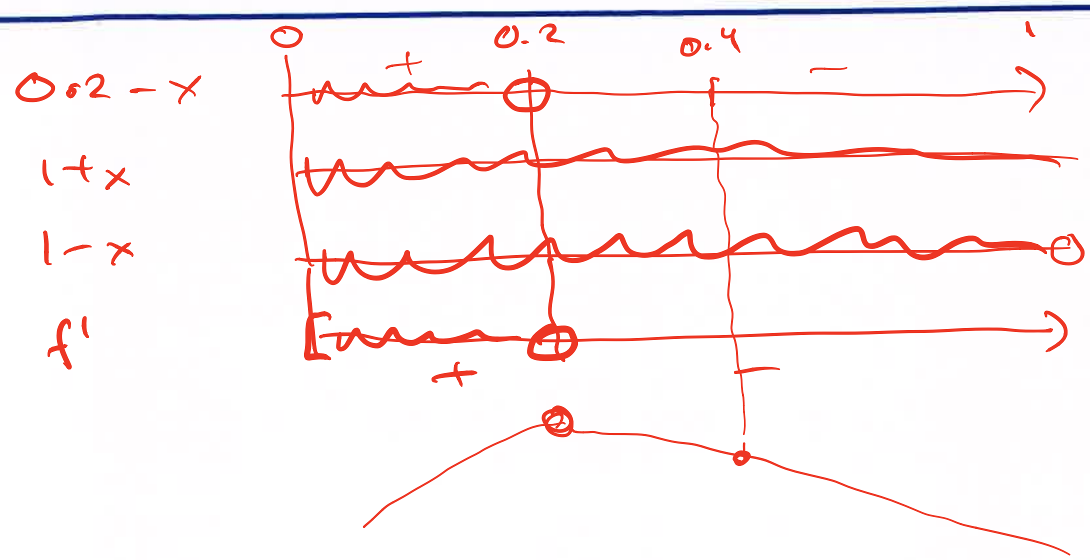
10. $f(x) = 0.6\ln(1+x)+0.4\ln(1-x)\,,\quad 0\le x<1$

a) max $f(x)$:

$$f'(x) = 0.6\cdot\frac{1}{1+x}+0.4\cdot\frac{1}{1-x}\cdot(-1)$$

$$= \boxed{\frac{0.6}{1+x}-\frac{0.4}{1-x}} = \frac{0.6(1-x)-0.4(1+x)}{(1+x)(1-x)}$$

$$= \frac{0.6-0.6x-0.4-0.4x}{(1+x)(1-x)} = \frac{0.2-x}{(1+x)(1-x)}$$



max: $x^* = \underline{0.2}$

$f(0.2) = 0.6 \cdot \ln 1.2 + 0.4 \ln 0.8$
$\simeq \underline{0.0201}$

b) $f'' > 0$: convex $\qquad$ $f'' < 0$: concave

$$f'(x) = \frac{0.6}{1+x} - \frac{0.4}{1-x} = 0.6 \cdot (1+x)^{-1} - 0.4\,(1-x)^{-1}$$

$$f''(x) = 0.6 \cdot (-1) \cdot (1+x)^{-2} \cdot 1 - 0.4 \cdot (-1)\,(1-x)^{-2} \cdot (-1)$$

$$= -\frac{0.6}{(1+x)^2} - \frac{0.4}{(1-x)^2} < 0 \quad \text{for all } x$$

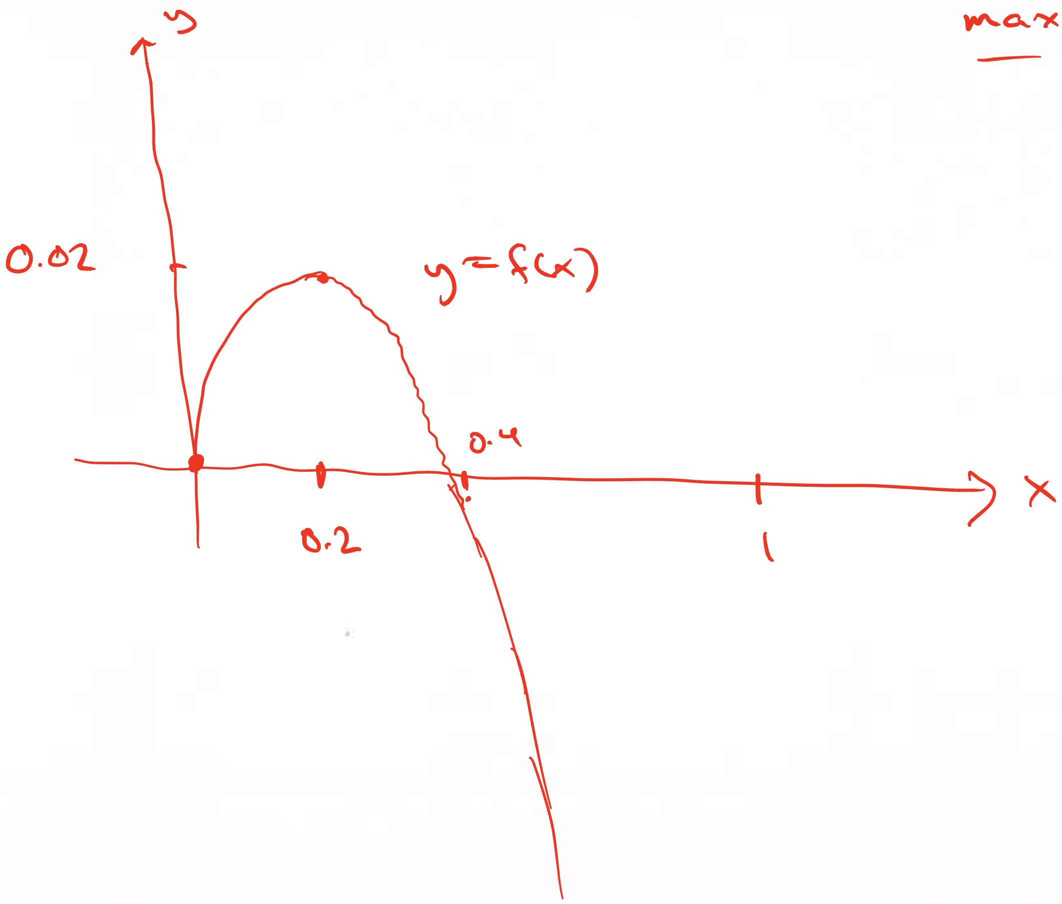
f is <u>concave</u>

c) <u>Show</u>: $f(x) < 0$ when $x > 2x^* = 0.4$

i) f is <u>decreasing</u> when $x \geq 0.4$ by b).
($f' < 0$)

ii) $f(0.4) = 0.6 \ln 1.4 + 0.4 \cdot \ln 0.6$
$\simeq -0.0024 < 0$

d) <u>Graph of f</u>:



max : $x = 0.2$

$f(0.2) \simeq 0.02$

$f(0) = 0$

$f(0.4) \simeq -0.0024$

f concave $\cap$