Plan

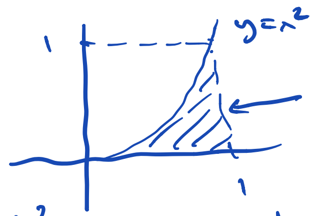
1 Definite integrals and antiderivation

Defn: $\int_a^b f(x)\,dx = \underline{F(b) - F(a)}$, where $F'(x) = f(x)$

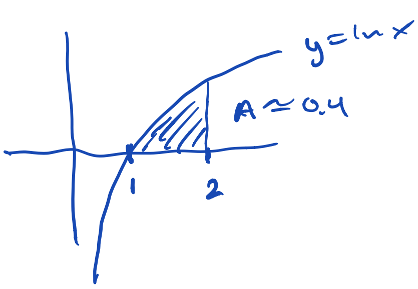
$= \left\{ \text{area under the graph of } f \text{ in } [a,b] \right\}$

Ex: $\int_0^1 x^2\,dx = \Big[\underbrace{\tfrac{1}{3}x^3 + C}_{F(x)}\Big]_0^1 = \underbrace{\left(\tfrac{1}{3}\cdot 1^3 + \cancel{C}\right)}_{F(1)} - \underbrace{\left(\tfrac{1}{3}\cdot 0^3 + \cancel{C}\right)}_{F(0)}$

$= \tfrac{1}{3} - 0 = \underline{\underline{1/3}}$



$\int_1^2 \ln x\,dx = \Big[x \ln x - x\Big]_1^2 = (2\ln 2 - 2) - (1 \cdot \cancel{\ln 1} - 1)$
$= \underline{2\ln 2 - 1} \simeq 0.4$



$\int \ln x\,dx = \int 1 \cdot \ln x\,dx =$ $\boxed{u = x \quad v = \ln x \quad u' = 1 \quad v' = 1/x}$

$= x \ln x - \int x \cdot 1/x\,dx$

$= x \ln x - \int 1\,dx = \underline{x \ln x - x + C}$

Ex:
$$\int_0^1 x\cdot\sqrt{x^2+1}\,dx = \int_1^2 x\sqrt{u}\cdot\frac{du}{2x} = \int_1^2 \frac{1}{2}u^{1/2}\,du$$

$\boxed{u=x^2+1,\quad du=2x\,dx}$ $\quad x=1 \Rightarrow u=2,\quad x=0 \Rightarrow u=1$

$$= \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_{u=1}^{u=2} = \underbrace{\left(\frac{1}{3}\cdot 2^{3/2}\right)}_{u=2} - \underbrace{\left(\frac{1}{3}\cdot 1^{3/2}\right)}_{u=1}$$

$$= \underline{\underline{\frac{1}{3}\cdot 2\sqrt{2}-\frac{1}{3}}} = \frac{1}{3}(2\sqrt{2}-1)$$

Alt:
$$\int_0^1 x\sqrt{x^2+1}\,dx = \int x\sqrt{u}\,\frac{du}{2x} = \int \frac{1}{2}u^{1/2}\,du = \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right].$$

$\boxed{u=x^2+1,\quad du=2x\,dx}$

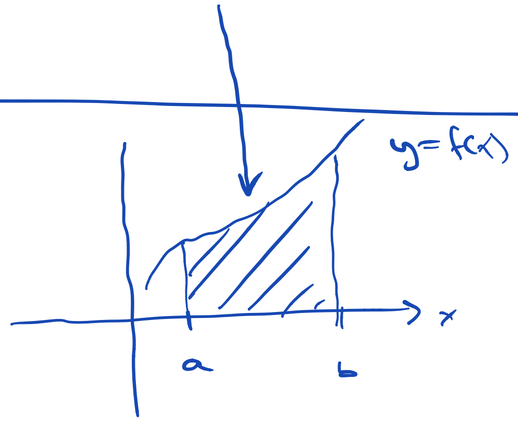
$$= \left[\frac{1}{3}(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}\cdot 2^{3/2} - \frac{1}{3}\cdot 1^{3/2}$$

$$= \frac{1}{3}\underline{\underline{(2\sqrt{2}-1)}}$$

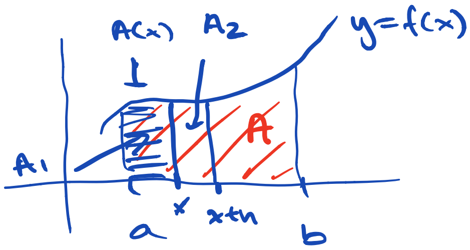
Theorem:
If f is a continuous function on $[a,b]$ and $f(x)\geq 0$ for $a\leq x\leq b$, then

$$\int_a^b f(x)\,dx = \left\{\begin{array}{l}\text{area under the}\\ \text{graph of } f \text{ in}\\ \text{the interval } [a,b]\end{array}\right\}$$

$F(b)-F(a)$
when $F'(x)=f(x)$



Explanation of the theorem:



Area function:

$$A(x) = \left\{ \begin{array}{l} \text{area under graph of} \\ f \text{ in } [a,x] \end{array} \right\}$$

for $a \leq x \leq b$

Facts:

$A(b) = A \qquad A(a) = 0$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \underset{h \text{ small}}{\simeq} \frac{A(x+h) - A(x)}{h}$$

$$= \frac{(A_1 + A_2) - A_1}{h} = \frac{A_2}{h}$$

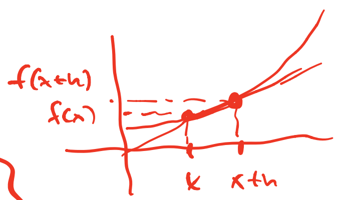
$$\simeq \frac{\cancel{h} \cdot f(x)}{\cancel{h}} = f(x)$$

$$\boxed{A'(x) = f(x)}$$

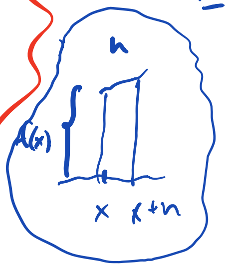
$A(x)$ antiderivative of $f(x)$

$f'(x)$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$



$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$



$$\int_a^b f(x)\,dx = \Big[A(x) \Big]_a^b$$

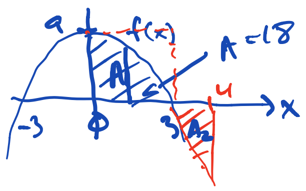
$$= A(b) - A(a) = A - 0 = \underline{\underline{A}}$$

area of strip $\hat{=} f(x) \cdot \Delta x$

$$\int_a^b f(x)\,dx$$

② Computing areas using definite integrals

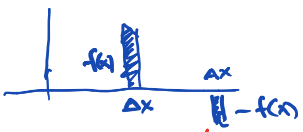
Ex: Area under $f(x) = 9 - x^2$ in $[0,3]$



$$\int_0^3 f(x)\,dx = A$$

$$A = \int_0^3 9 - x^2\,dx = \Big[9x - \tfrac{1}{3}x^3\Big]_0^3$$

$$= (9 \cdot 3 - \tfrac{1}{3} \cdot 3^3) - (0) = 27 - 9 = \underline{\underline{18}} \leftarrow A_1$$



Ex: $$\int_0^4 9 - x^2\,dx = \Big[9x - \tfrac{1}{3}x^3\Big]_0^4 = (9 \cdot 4 - \tfrac{1}{3} \cdot 4^3) - 0$$

$$= 36 - \frac{64}{3} = \frac{108 - 64}{3} = \frac{44}{3} \simeq \underline{14{,}67} \leftarrow \underset{18}{\overset{\shortparallel}{A_1}} - \underset{3{,}33}{\overset{\shortparallel}{A_2}}$$

Computing areas: Assume $f(x)$ cont on $[a,b]$

i) $f(x) \geq 0$ in $[a,b]$: $\boxed{\int_a^b f(x)\,dx = A}$ ← height $f(x) - 0 = f(x)$

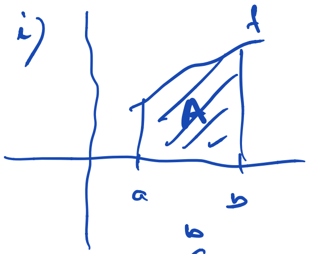
ii) $f(x) \leq 0$ — " — : $\int_a^b f(x)\,dx = -A \iff \boxed{A = \int_a^b -f(x)\,dx}$

height: $0 - f(x) = -f(x)$

iii) $f(x) \geq g(x)$ — " — : $\boxed{A = \int_a^b f(x) - g(x)\,dx}$

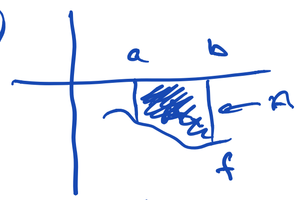
height $= f(x) - g(x)$

i)



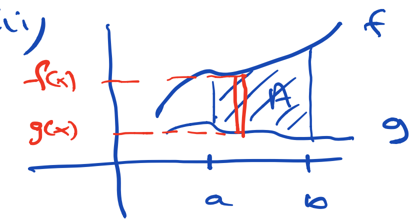
$A = \int_a^b f(x)\,dx$

ii)



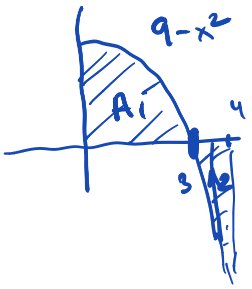
$A = \int_a^b -f(x)\,dx$

iii)



$A = \int_a^b f(x) - g(x)\,dx$

Ex: Area between $f(x) = 9 - x^2$ and the x-axis in $[0,4]$.



$A = A_1 + A_2$

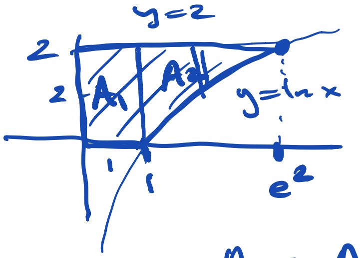
$$A = A_1 + A_2 = \int_0^3 9 - x^2\,dx + \int_3^4 \underbrace{-(9-x^2)}_{\text{height} = 0 - f(x)}\,dx$$

$$= 18 + \left[-9x + \tfrac{1}{3}x^3\right]_3^4$$

$$= 18 + \left(-9 \cdot 4 + \tfrac{1}{3} \cdot 4^3\right) - \left(-27 + \tfrac{1}{3} \cdot 3^3\right)$$

$$= 18 - 36 + 27 + \tfrac{64}{3} - 9 = 64/3 \simeq \underline{\underline{21{,}3}}$$

Ex: Compute the area bounded by $y = \ln(x)$, the line $y = 2$, the x-axis and the y-axis.



Intersections:

$y = \ln(x)$ and x-axis ($y=0$)

$\ln x = 0$ | $e^{\cdot}$

$x = e^0 = 1$

$y = \ln x$ and $y = 2$

$\ln x = 2$ | $e^{\cdot}$

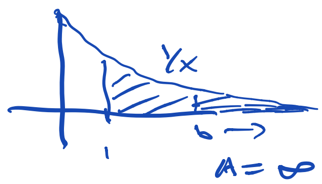
$x = e^2$

$$A = A_1 + A_2$$
$$= 1 \cdot 2 + \int_1^{e^2} 2 - \ln x \, dx$$
$$= 2 + \left[2x - (x \ln x - x) \right]_1^{e^2}$$
$$= 2 + \left[3x - x \ln x \right]_1^{e^2}$$
$$= 2 + \left(3e^2 - e^2 \cdot \underbrace{\ln e^2}_{2}\right) - \left(3 - 1 \cdot \underbrace{\ln 1}_{0}\right)$$
$$= 2 + 3e^2 - 2e^2 - 3 = \underline{\underline{e^2 - 1}}$$

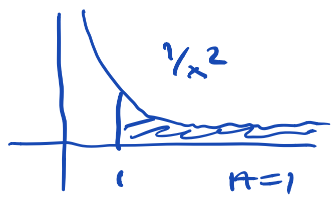
③ Improper integrals:

Ex: $\int_1^\infty \frac{1}{x} \, dx \qquad \int_1^\infty \frac{1}{x^2} \, dx \qquad \int_0^1 \frac{1}{x} \, dx$

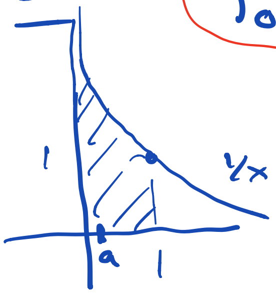
$$\int_1^\infty \frac{1}{x} \, dx = \left[\ln |x| \right]_1^\infty = \lim_{b \to \infty} \left[\ln |x| \right]_1^b$$
$$= \lim_{b \to \infty} \left(\ln b - \underbrace{\ln 1}_{0} \right) = \underline{\underline{\infty}}$$



$$\int_1^\infty \frac{1}{x^2} dx = \int_1^\infty x^{-2} dx = \left[-\frac{1}{x} \right]_1^\infty$$
$$= \lim_{b \to \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right)$$
$$= \underline{\underline{1}}$$



Ex:



$$\int_0^1 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_0^1 = \lim_{a\to 0^+}\Big[\ln|x|\Big]_a^1$$

$$= \lim_{a\to 0^+}\big(\underbrace{\ln 1}_{=0} - \ln a\big) = \lim_{a\to 0^+}(-\ln a) = \infty$$

$f(x) = 1/x$ is not defined on $[0,1]$
$(x = 0)$

Part 2:

* $\int_a^b f(x)\,dx = F(b)-F(a)$, $F'(x)=f(x)$

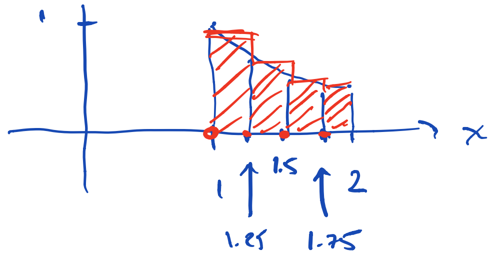
* $\int_a^b f(x)\,dx = A$ when $f(x)\geq 0$ in $[a,b]$

$A = \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ " $f(x)\leq 0$ — " —

$\boxed{A = \int_a^b f(x)-g(x)\,dx}$ " $f(x)\geq g(x)$ — " —

Problems:

3. b) $n=4$ $\int_1^2 \frac{1}{x}\,dx$



$0.25\cdot f(1) + 0.25\cdot f(1.25) + 0.25\cdot f(1.5) + 0.25\cdot f(1.75)$

$= 0.25\left(\frac{1}{1}+\frac{1}{1.25}+\frac{1}{1.5}+\frac{1}{1.75}\right) = \frac{1}{4}\cdot\left(\frac{4}{4}+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}\right)$

$\uparrow 4/4 \quad \uparrow 5/4 \quad \uparrow 6/4 \quad \uparrow 7/4$

$= \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$

$\simeq 0.25 + 0.20 + 0.167 + 0.14$

$\simeq 0.57$

8. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n-1}\right) = \ln(2)$

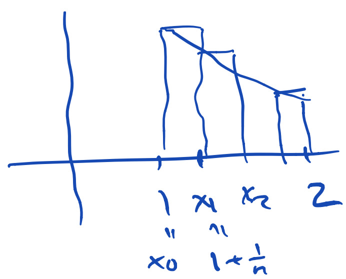
① Area: $\int_1^2 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^2 = \ln 2 - \ln 1 = \ln 2$

② Riemann sum, n subintervals:

$\Delta x = \frac{2-1}{n} = 1/n \qquad x_i = 1 + i\cdot\frac{1}{n}$

$\Delta x\cdot f(x_0) + \Delta x\cdot f(x_1) + \dots + \Delta x\cdot f(x_{n-1})$

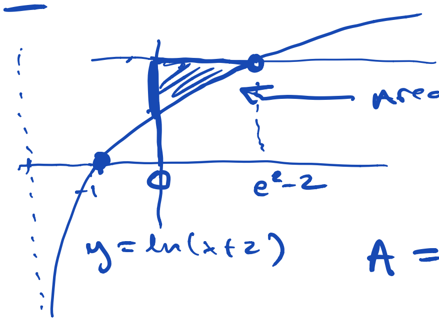
$= \frac{1}{n}\left(\frac{1}{1}+\frac{1}{1+1/n}+\frac{1}{1+2/n}+\dots+\frac{1}{2-1/n}\right)$



$$= \frac{1}{\cancel{n}}\cdot\left(\frac{\cancel{n}}{n}+\frac{\cancel{n}}{n+1}+\frac{\cancel{n}}{n+2}+\dots+\frac{\cancel{n}}{2n-1}\right)$$

$$= \frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n-1} \xrightarrow[n\to\infty]{} A = \ln 2$$

5. R: Bounded by $y=\ln(2+x)$, $y=2$, y-axis



$$A = \int_0^{e^2-2} 2 - \ln(2+x)\,dx$$

Intersection:

$y=\ln(x+2)$ and $y=2$

$\ln(x+2)=2 \quad |e$

$x+2=e^2$

$x=e^2-2$

$$= \Big[2x - \big(x\ln|x+2| - x + 2\ln(x+2)\big)\Big]_0^{e^2-2}$$

$$= \Big[3x - x\ln|x+2| - 2\ln(x+2)\Big]_0^{e^2-2}$$

$$= \Big[3x - (x+2)\ln(x+2)\Big]_0^{e^2-2}$$

$$= \Big(3(e^2-2) - e^2\underbrace{\ln(e^2)}_{2}\Big) - \big(0 - 2\ln 2\big)$$

$$= 3e^2 - 6 - 2e^2 + 2\ln 2$$

$$= \underline{\underline{e^2 - 6 + 2\ln 2}}$$

$$\int 1\cdot\ln(2+x)\,dx \qquad \boxed{\begin{array}{ll} u=x & v=\ln(2+x) \\ u'=1 & v'=\frac{1}{2+x}\end{array}}$$

$$= x\cdot\ln(2+x) - \int \frac{x}{x+2}\,dx$$

$$= x\cdot\ln(x+2) - \int \frac{(x+2)-2}{x+2}\,dx$$

$$= x\ln(x+2) - \int 1 - \frac{2}{x+2}\,dx$$

$$= x\ln(x+2) - x + 2\ln|x+2| + C$$

7d.

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_0^1 \frac{e^u}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du$$

$$\boxed{\begin{array}{l} u=\sqrt{x} \\ du=\frac{1}{2\sqrt{x}}\,dx\end{array}}$$

$x=0 \Rightarrow u=0$

$x=1 \quad\;\; u=1$

$$= \int_0^1 2e^u\,du = \Big[2e^u\Big]_0^1 = 2e - 2\cdot 1 = \underline{\underline{2e-2}}$$

e. $\int_0^\infty \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^{-\infty} \frac{e^u}{\cancel{\sqrt{x}}} \cdot (-2\cancel{\sqrt{x}})\,du$

$\boxed{\begin{aligned} u &= 1-\sqrt{x} \\ du &= -\frac{1}{2\sqrt{x}}\,dx \end{aligned}}$ $\quad x=0: \; u=1$
$\quad x=\infty: \; u=-\infty$

$$= \int_1^{-\infty} -2e^u\,du = \left[-2e^u\right]_1^{-\infty} = \lim_{b\to-\infty} \left[-2e^u\right]_1^b$$

$$= \lim_{b\to-\infty} \left(-2e^b + 2e\right) = 2e$$

$$\left(b \to -\infty \Rightarrow e^b = \frac{1}{e^{-b}}\right)$$