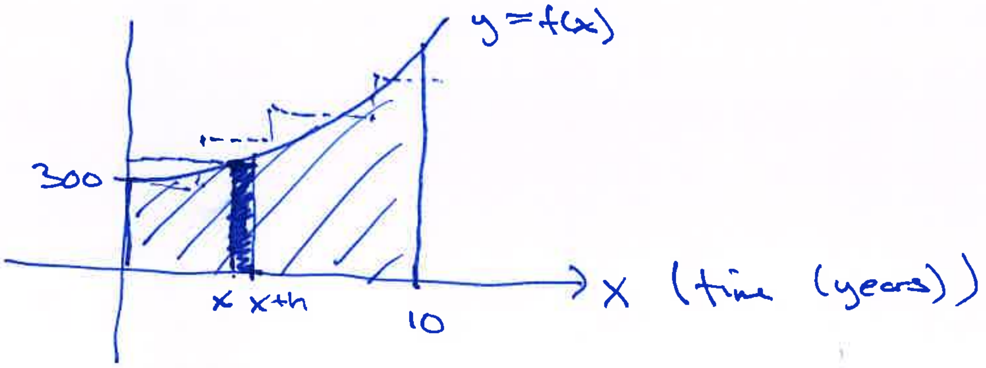
Plan

1 Economic application of integration
2 Examples

① Economic applications

a) Cash flows (continuous)

Ex: $f(x) = 300 \cdot 1.06^x$ (cash flow rate, MNOK / year)



Total cash flow in the first 10 years:

$$\int_0^{10} f(x)\,dx = \int_0^{10} 300 \cdot 1.06^x\,dx$$

$$= \left[300 \cdot 1.06^x \cdot \frac{1}{\ln(1.06)}\right]_0^{10}$$

$$= \left[\frac{300}{\ln(1.06)} \cdot 1.06^x\right]_0^{10}$$

$$= \frac{300}{\ln 1.06} \cdot 1.06^{10} - \frac{300}{\ln 1.06} \cdot 1$$

$$= \frac{300}{\ln(1.06)} \cdot (1.06^{10} - 1) \simeq 4.072$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = a^x \cdot \frac{1}{\ln(a)} + C$$

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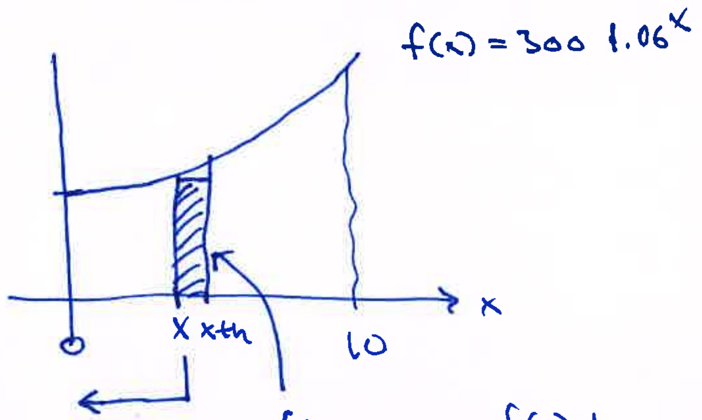
Net present value of cash flows:



$$f(x)\cdot \Delta x = f(x)\cdot h$$

$$\frac{f(x)\cdot\Delta x}{e^{rx}} = f(x)\cdot\Delta x\cdot e^{-rx} \leftarrow$$ present value of the cash flow in the period $[x, x+h]$

Use continuous discounting:

- e^{-rx}

Discount rate: $r = 0.10$

Formula, NPV of cash flow:

$$\int_0^{10} f(x)\cdot e^{-rx}\,dx$$

$$1.06^x = \left(e^{\ln 1.06}\right)^x = e^{\ln(1.06)\cdot x}$$

$$NPV = \int_0^{10} 300\cdot 1.06^x\, e^{-0.10x}\,dx$$

$$= \int_0^{10} 300\cdot e^{\ln(1.06)x}\cdot e^{-0.10x}\,dx$$

$$= \int_0^{10} 300\, e^{\ln(1.06)x - 0.10x}\,dx$$

$u = \ln(1.06)x - 0.1x$

$du = (\ln 1.06 - 0.1)\,dx$

$$= \int_0^{10\cdot\ln(1.06)-1} 300\, e^u \cdot \frac{du}{\ln 1.06 - 0.10}$$

$$= \int_0^{10\ln(1.06)-1} \frac{300}{\ln 1.06 - 0.10}\, e^u\,du = \left[\frac{300}{\ln 1.06 - 0.10}\, e^u + C\right]_0^{10\ln 1.06 - 1}$$

$$= \frac{300}{\ln 1.06 - 0.10}\cdot e^{10\cdot\ln 1.06 - 1} - \frac{300}{\ln 1.06 - 0.10}\cdot 1$$

$$= \frac{300}{\ln 1.06 - 0.10}\left(\frac{1.06^{10}}{e} - 1\right) \simeq 2.453$$

$\simeq -0.04$

$\simeq -0.34$

$\simeq -7188.9$ $\simeq 0.16$

Summary:

$f(x)$: cash flow rate

r : discount rate

$[t_1, t_2]$: time interval

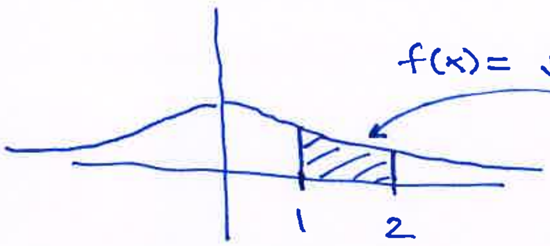
Total cash flow: $\int_{t_1}^{t_2} f(x)\,dx$

Net present value of cash flow: $\int_{t_1}^{t_2} f(x) \cdot e^{-rx}\,dx$

b) probabilities

when the probability density function

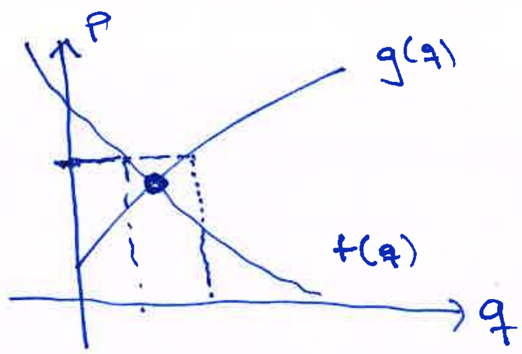
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



$P(1 \le X \le 2) = \int_1^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

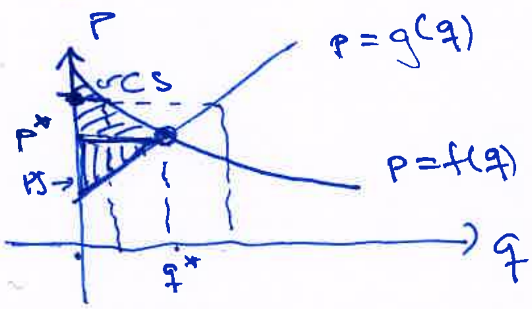
Standard normal distribution

c) Consumer/producer surplus



$p = f(q)$ inverse demand fn.

$p = g(q)$ inverse supply fn.



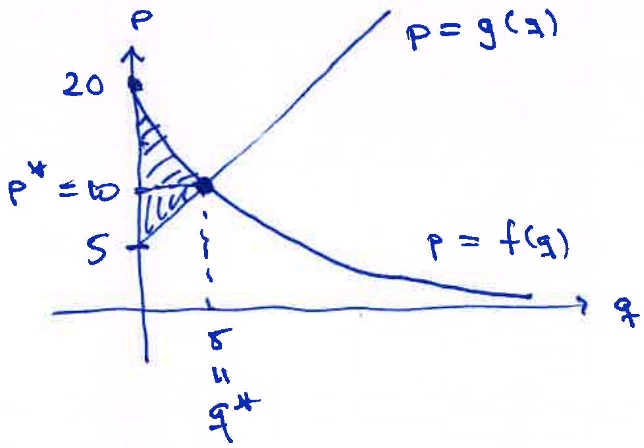
CS = consumer surplus

$= \int_0^{q^*} f(q) - p^*\,dq$

PS = producer surplus

$= \int_0^{q^*} p^* - g(q)\,dq$

Ex: $f(q) = \frac{100}{q+5}$ $g(q) = q+5$



Intersection = marked cross

$f(q) = g(q)$

$\frac{100}{q+5} = q+5$

$100 = (q+5)^2$

$\pm\sqrt{100} = q+5$

$q+5 = \pm 10$

$q = \pm 10 - 5$

$\underline{q=5}$ or $\cancel{q=-15}$

$$CS = \int_0^5 \frac{100}{q+5} - 10 \, dq$$

$$= \left[100 \ln(q+5) - 10q \right]_0^5$$

$$= (100 \ln 10 - 50) - (100 \cdot \ln 5)$$

$$= 100 (\ln 10 - \ln 5) - 50$$

$$= 100 \ln\left(\frac{10}{5}\right) - 50 = 100 \ln 2 - 50$$

$$\simeq \underline{\underline{19}}$$

$$PS = \int_0^5 10 - (q+5) \, dq = \int_0^5 5 - q \, dq = \left[5q - \tfrac{1}{2} q^2 \right]_0^5$$

$$= \left(5 \cdot 5 - \tfrac{1}{2} \cdot 5^2\right) - 0 = 25 - \frac{25}{2} = \frac{25}{2} = \underline{\underline{12.5}}$$

② Examples : Problem set 19, pb 9

a) $\int 30\, x\sqrt{x}\, dx = \int 30\, x^{3/2}\, dx$

$n = 3/2$
$n+1 = 5/2$

$= 30 \cdot \dfrac{x^{5/2} \cdot 2}{5\cancel{x} \cdot \cancel{2}} + C$

$= \dfrac{60}{5} x^2 \cdot \sqrt{x} + C = 12\, x^2 \sqrt{x} + C$

b) $\int x e^{-x} dx = -e^{-x} \cdot x - \int -e^{-x} \cdot 1\, dx$

$$\boxed{\begin{array}{ll} u = -e^{-x} & v = x \\ u' = e^{-x} & v' = 1 \end{array}}$$

$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C$

$$\int e^{-x} dx = \int e^{u} \cdot \frac{du}{-1} = \int -e^{u} du$$

$u = -x$
$du = -1 \cdot dx$

$= -e^{u} + C = -e^{-x} + C$

c) $\int \dfrac{6-3x}{4-9x^2} dx$

Partial fractions:

$\dfrac{6-3x}{4-9x^2} = \dfrac{6-3x}{(2+3x)(2-3x)} = \dfrac{A}{2+3x} + \dfrac{B}{2-3x} \quad \Big| \cdot (2+3x)(2-3x)$

$4 - 9x^2 = 0 \qquad x = \pm 2/3$
$9x^2 = 4$
$x^2 = 4/9 \qquad -9(x-2/3)(x+2/3)$

$6 - 3x = A(2-3x) + B(2+3x)$

$6 - 3x = (2A + 2B) + (-3A + 3B)x$

$2A+2B = 6 \qquad -3A+3B = -3$

$A+B=3$

$A-B=1$

$2A \quad =4$

$A=2$

$B=1$

$$\int \frac{6-3x}{4-9x^2}\,dx = \int \frac{2}{2+3x} + \frac{1}{2-3x}\,dx$$

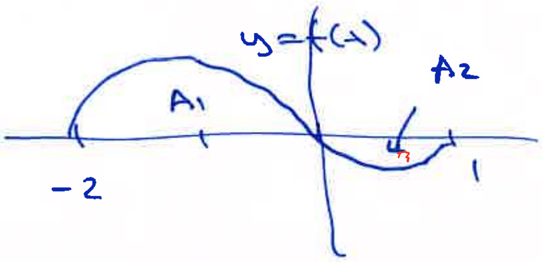
$$= \frac{2}{3}\cdot \ln|2+3x| + \left(\frac{1}{-3}\right)\ln|2-3x| + C$$

$$= \frac{2}{3}\ln|2+3x| - \frac{1}{3}\ln|2-3x| + C$$

$$= \frac{1}{3}\left(2\ln|2+3x| - \ln|2-3x|\right) + C$$

$$= \frac{1}{3}\ln\frac{|2+3x|^2}{|2-3x|} + C$$

d)



$A_2 = 22/15$

$$\int_{-2}^{1} f(x)\,dx = \frac{18}{5}$$

$$+A_1 - A_2 = \frac{18}{5}$$

Find A_1:

$$A_2 = \int_0^1 -f(x)\,dx \iff \int_0^1 f(x)\,dx = -A_2$$

$$\underbrace{\int_{-2}^{0} f(x)\,dx}_{A_1} + \underbrace{\int_0^1 f(x)\,dx}_{-A_2}$$

$$A_1 - \frac{22}{15} = \frac{18}{5}$$

$$A_1 = \frac{22}{15} + \frac{18\cdot 3}{5\cdot 3}$$

$$= \frac{22+54}{15} = \frac{76}{15}$$

Part 2:

$$\int_{t_1}^{t_2} f(x)\,dx = \text{total cash flow in } [t_1, t_2] \qquad f(x): \text{cash flow rate}$$

$$\int_{t_1}^{t_2} f(x) e^{-rx}\,dx = \text{net present value of cash flow} \qquad r: \text{disc. rate}$$

Problems Problem set 20

5. a) $\displaystyle\int \frac{3x-4}{x^2+x}\,dx = \int \frac{-4}{x} + \frac{3}{x+1}\,dx = \underline{-4\ln|x| + 3\ln|x+1| + C}$

$$\frac{3x-4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Big| \cdot x(x+1)$$

$$3x-4 = A(x+1) + Bx$$

$$3x-4 = (A+B)x + A$$

$A+B = 3$, $A = -4$

$A = -4$

$B = 7$

$$= \ln|x+1|^3 - \ln|x|^4 + C$$

$$= \underline{\ln\frac{|x+1|^3}{|x|^4} + C}$$

b) $\displaystyle\int \underbrace{18x^2}_{u'} \underbrace{\ln(x+1)}_{v}\,dx = 6x^3\ln(x+1) - \int 6x^3 \cdot \frac{1}{x+1}\,dx$

$u = 6x^3$ $\quad v = \ln(x+1)$

$u' = 18x^2$ $\quad v' = \frac{1}{x+1}$

$$= 6x^3\ln(x+1) - \int \frac{6x^3}{x+1}\,dx = 6x^3\ln(x+1) - \int 6x^2 - 6x + 6 - \frac{6}{x+1}\,dx$$

$$6x^3 : (x+1) = 6x^2 - 6x + 6$$
$$-(6x^3 + 6x^2)$$
$$-6x^2$$
$$-(-6x^2 - 6x)$$
$$6x$$
$$-(6x+6)$$
$$-6$$

$$= \underline{6x^3\ln(x+1) - 2x^3 + 3x^2 - 6x - 6\ln|x+1| + C}$$

c) $\int e^{\sqrt{x}}\,dx = \int e^u (2\sqrt{x})\,du = \int \underbrace{2u}_{w}\,\underbrace{e^u}_{v'}\,du$

$\boxed{u=\sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}\,dx}$

$= 2ue^u - \int e^u \cdot 2\,du$

$= 2ue^u - 2e^u + C$

$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

$\int v'w\,du = vw - \int vw'\,du$

$\boxed{\begin{matrix} v = e^u & w = 2u \\ v' = e^u & w' = 2 \end{matrix}}$

d) $\int_1^{\infty} \frac{1}{x^2+x}\,dx = \int_1^{\infty} \frac{1}{x} - \frac{1}{x+1}\,dx = \Big[\ln|x| - \ln|x+1|\Big]_1^{\infty}$ "$\infty - \infty$"

$= \lim_{b\to\infty} \left[\ln \frac{|x|}{|x+1|}\right]_1^b$

$= \lim_{b\to\infty} \ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{1}{2}\right) = -\ln(1/2)$

$\frac{b}{b+1} \to 1$

$= -\ln(2^{-1})$

$= \ln 2 \approx 0.69$

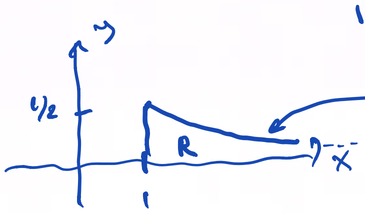
$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x+1) + Bx$

$1 = (A+B)x + A$

$A+B = 0, \quad A = 1$

$B = -1$



6. $V(t) = 120\,e^{\sqrt{t}/5}$

a) $\max\ 120\,e^{\sqrt{t}/5} \cdot e^{-0.04t} = \max\ \underbrace{120\,e^{\sqrt{t}/5 - 0.04t}}_{N(t)}$, $u = \sqrt{t}/5 - 0.04t$

$N'(t) = 120\,e^u \cdot u' = 120\,e^u \cdot \left(\frac{1}{5}\cdot\frac{1}{2\sqrt{t}} - 0.04\right)$

$= 120\,e^u \left(\frac{1\cdot 10}{10\sqrt{t}\cdot 10} - \frac{4\sqrt{t}}{100\sqrt{t}}\right) = 120\,e^u\ \frac{10-4\sqrt{t}}{100\sqrt{t}}$

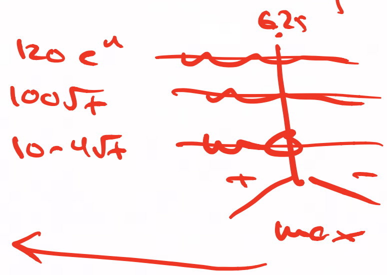
pos pos

$10 - 4\sqrt{t} = 0$

$4\sqrt{t} = 10$

$\sqrt{t} = \frac{10}{4} = \frac{5}{2}$

$t = \frac{25}{4} = 6.25$



b)

$V(t) = 2 \cdot 120$

$\cancel{120} e^{\sqrt{t}/5} = 2 \cdot \cancel{120}$

$\sqrt{t}/5 = \ln 2$

$\sqrt{t} = 5 \ln 2$

$t = \underline{5^2 (\ln 2)^2}$

$T \approx \underline{\underline{12 \text{ years}}}$

$V(t) = 4 \cdot 120$

$\cancel{120} e^{\sqrt{t}/5} = 4 \cdot \cancel{120}$

$\sqrt{t}/5 = \ln 4 = \ln 2^2 = 2 \ln 2$

$\sqrt{t} = 5 \ln 4 = 5 \cdot 2 \ln 2$

$t = 5^2 \cdot (\ln 4)^2$

$= 5^2 \cdot 2^2 \cdot (\ln 2)^2$

$= \underline{\underline{4T}} \approx 48 \text{ years}$