Plan

1 Linear systems of equations
2 Gaussian elimination

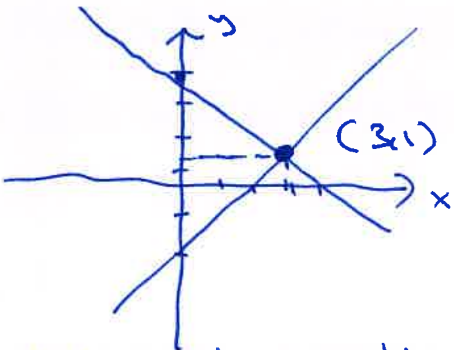
Lectures: Friday 10 – 12
Problem Session: 12 – 15

Exams:
Home exam, pass/fail.

① Linear systems of equations.

Ex:
$$\boxed{\begin{matrix} x+y=4 \\ x-y=2 \end{matrix}} \quad \begin{matrix} \to y=4-x \\ \to y=x-2 \end{matrix}$$

2x2 linear



Substitution method:

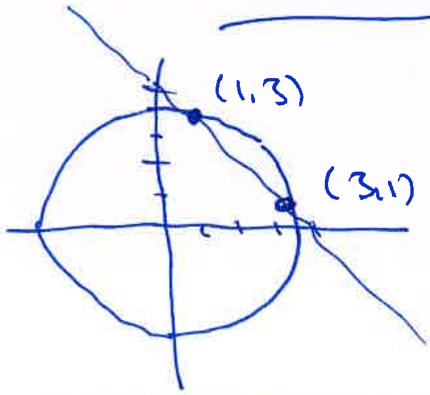
$$\boxed{\begin{matrix} x+y=4 \\ x-y=2 \end{matrix}} \to y=4-x$$

$$x-y=2$$
$$x-(4-x)=2$$
$$2x-4=2$$
$$2x=6$$
$$x=3$$
$$y=1$$

Solution: $(x,y)=(3,1)$

Ex:
$$x^2+y^2=10 \quad \text{circle } r=\sqrt{10}$$
$$x+y=4$$

not linear



$$\boxed{\begin{matrix} x^2+y^2=10 \\ x+y=4 \end{matrix}} \to y=4-x$$

$$x^2+y^2=10$$
$$x^2+(4-x)^2=10$$
$$x^2+16-8x+x^2=10$$
$$2x^2-8x+6=0 \quad |:2$$
$$x^2-4x+3=0$$

$x=3$ or $x=1$
$y=1$ $\quad$ $y=3$

Solution:
$(x,y)=(3,1),(1,3)$

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m×n linear system:

linear system of equations with m equations in n variables (all equations are linear)

Ex:
$$x+y+z=3 \quad (1)$$
$$x+2y+4z=7 \quad (2)$$
$$x+3y+9z=13 \quad (3)$$

3×3 linear system

Elimination method:

$$\begin{aligned} x+y+z&=3 && \text{I}\\ x+2y+4z&=7 && \text{II}\\ x+3y+9z&=13 && \text{III}\end{aligned}$$

↓

$$\begin{aligned} &\text{A} & x+y+z&=3 && \text{I}\\ &\text{B} & y+3z&=4 && \text{II}-\text{I}\\ &\text{C} & 2y+8z&=10 && \text{III}-\text{I}\end{aligned}$$

↓

$$\begin{aligned} &\text{A} & x+y+z&=3\\ &\text{B} & y+3z&=4\\ &\text{C}-2\text{B} & 2z&=2\end{aligned}$$

Back substitution:

$2z=2 \Rightarrow z=1$

$y+3z=4 \quad y=1$

$x+y+z=3 \quad x=1$

Substitution:

(1) $\boxed{x=3-y-z}$

(2) $(3-y-z)+2y+4z=7$

$\boxed{y+3z=4}$

(3) $(3-y-z)+3y+9z=13$

$\boxed{2y+8z=10}$

$$\boxed{\begin{aligned} y+3z&=4\\ 2y+8z&=10\end{aligned}} \rightarrow y=4-3z$$

$2(4-3z)+8z=10$

$2z=2$

$z=1$

$y=1$

$x=1$

Solution:

$(x,y,z)=(1,1,1)$

Ex:

$$\boxed{\begin{aligned} x+3y&=7\\ 2x-y&=0\end{aligned}} \quad \cdot 2$$

↓

$$\begin{aligned} 2x+6y&=14\\ -\quad 2x-y&=0\\ \hline =\quad 7y&=14\end{aligned}$$

↓

$y=2,\ x=1$

$$\begin{aligned} x+y&=4\\ +\quad x-y&=2\\ \hline =\quad 2x&=6 \Rightarrow x=3\end{aligned}$$

$y=1$

$$\begin{aligned} x+y&=4\\ -\quad x-y&=2\\ \hline =\quad 2y&=2\end{aligned}$$

$y=1$

$x=3$

(2) Gaussian elimination

m×n linear system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}\cdot x_n = b_m \end{array}\right\} \text{ m equations}$$

n variables

$x_1, x_2, \dots, x_n$

a_{ij}, b_j : given numbers

a_{12} : coeff in equ. 1 of x_2

b_2 : const. in second equation

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

coefficient matrix

$$(A|\underline{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

extended matrix (augmented)

Ex:

$$\begin{array}{l} x+y+z=3 \\ x+2y+4z=7 \\ x+3y+9z=13 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

extended matrix

Gaussian elimination : method for solving any linear system

linear system → extended matrix

↓ elementary row operation

echelon form → (simpler) linear system

back substitution → solutions

Ex:

$$\begin{aligned} x+2y-z&=3\\ 3x-y+z&=6\\ 2x+3y-4z&=3 \end{aligned}$$

3x3 linear system

Elementary row operations:

i) switch two rows
ii) multiply a row by $c \neq 0$
iii) add a multiple of one row to another row

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 1 & 6 \\ 2 & 3 & -4 & 3 \end{array}\right) \cdot(-3) \leftarrow R(2)+(-3)\cdot R(1) \to R(2)$$

$$\downarrow$$

pivot: first non-zero element in a row

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 4 & -3 \\ 2 & 3 & -4 & 3 \end{array}\right) -2$$

Echelon form: A matrix s.t.

i) All zero rows are below non-zero rows.
ii) All entries under a pivot are zero.

$$\downarrow$$

$$-\tfrac{1}{7}\cdot \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 4 & -3 \\ 0 & -1 & -2 & -3 \end{array}\right) \updownarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 4 & -3 \\ 0 & 0 & -18/7 & -18/7 \end{array}\right)$$

echelon form

$-2-4/7 = -14/7 - 4/7 = -18/7$

$-3+3/7 = -21/7 + 3/7 = -18/7$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & -7 & 4 & -3 \end{array}\right) -7$$

$$\downarrow$$

$$\begin{array}{cccc} x & y & z & = \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 18 & 18 \end{array}\right)$$

echelon form

$$\begin{aligned} x+2y-z&=3\\ -y-2z&=-3\\ 18z&=18 \end{aligned}$$

Back substitution:

$x+2\cdot 1-1=3 \Rightarrow x=2$

$-y-2\cdot 1=-3 \Rightarrow -y=-1 \Rightarrow y=1$

$18z=18 \Rightarrow z=1$

Solution: $(x,y,z)=(2,1,1)$

Ex:

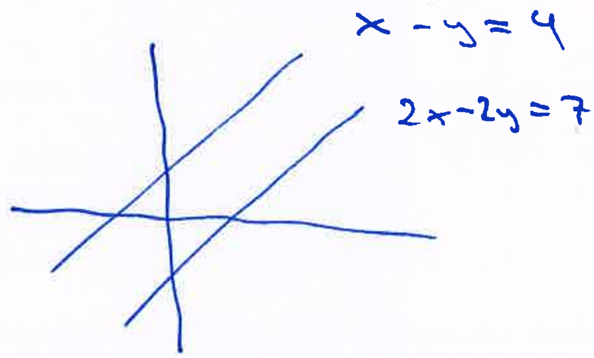
$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

echelon form

$$x + 2y + 3z = 4$$
$$0 = -1$$

impossible

In general: $\left\{ \text{pivot in the last column} \right\} \Longleftrightarrow$ no solutions (inconsistent)



Ex:

$$\begin{array}{c|c} \begin{array}{cccc} x & y & z & w \end{array} & \\ \left(\begin{array}{cccc|c} 1 & 1 & -1 & 4 & 7 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) & \\ \begin{array}{cccc} - & - & \times & - \end{array} & \end{array}$$

echelon form

① there are solutions (consistent)

Back substitution:

$$x + y - z + 4w = 7$$
$$y + w = 5$$
$$w = 2$$

z free

$$x = 7 - 3 + z - 8 = -4 + z$$
$$y = 3 \checkmark$$
$$w = 2 \checkmark$$

$$\boxed{(x, y, z, w) = (z - 4, 3, z, 2) \text{ with } z \text{ free}}$$

When there are free variables, there are infinitely many solutions.

Part 2:

Gaussian elimination

linear system → extended matrix ↓ elementary row op. → echelon form → system of eqn.s ↓ back substitution → solution

pivot in the last column — no solutions (inconsistent)

no pivot in the last column — there are solutions (consistent)

- pivot in every variable column: all variables are basic (can be solved for) ⇓ one solution
- some variable columns don't have pivots: {variable columns without pivots} ⟷ free variables ⇓ infinitely many solutions

1d

$$x^2 - y^2 = 8$$
$$xy = 3 \Rightarrow y = 3/x \Rightarrow x^2 - (3/x)^2 = 8$$
$$x^2 - 9/x^2 = 8 \quad | \cdot x^2$$
$$x^4 - 9 = 8x^2$$
$$x^4 - 8x^2 - 9 = 0$$

$u = x^2$: $u^2 - 8u - 9 = 0$

$u = 9$ or $u = -1$

$x^2 = 9$ or $\cancel{x^2 = -1}$

$x = \pm 3 \Rightarrow y = \pm 1$

Solutions:

$(x, y) = (3, 1), (-3, -1)$

3b

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 2 & -4 & 1 & 1 \\ 3 & -5 & 2 & 4 \end{array}\right) \begin{matrix} \\ -2 \\ -3 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & -2 & -1 & -5 \\ 0 & -2 & -1 & -5 \end{array}\right) \begin{matrix} \\ \\ -1 \end{matrix}$$

$$\begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

echelon form

z free

infinitely many solutions

$$\underline{x} - y + z = 3$$

$$-\underline{2y} - z = -5$$

$$-2y = z - 5$$

$$y = \underline{-z/2 + 5/2}$$

Solutions:

$$(x,y,z) = \left(11/2 - 3z/2,\ 5/2 - z/2,\ z\right)$$

(with z free)

$$x = 3 + y - z$$

$$= 3 + (-z/2 + 5/2) - z$$

$$= \underline{-3z/2 + 11/2}$$

6b

$$\left(\begin{array}{ccc|c} 3 & 4 & 3 & 2 \\ 2 & -1 & 1 & 1 \\ 7 & 2 & 5 & 3 \end{array}\right) \begin{matrix} -1 \\ \\ \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 5 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 7 & 2 & 5 & 3 \end{array}\right) \begin{matrix} \\ -2 \\ -7 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 5 & 2 & 1 \\ 0 & -11 & -3 & -1 \\ 0 & -33 & -9 & -4 \end{array}\right) \begin{matrix} \\ \\ -3 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 5 & 2 & 1 \\ 0 & -11 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

no solutions

7.

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 1 & 2 & 4 & -1 & 7 \\ 1 & -1 & 1 & 11 & 16 \end{array}\right) \begin{matrix} \\ ]{\cdot}(-1) \\ ]{\cdot}(-1) \end{matrix}$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & -2 & -3 \\ 0 & -2 & 0 & 10 & 6 \end{array}\right) \begin{matrix} \\ \\ ]+2 \end{matrix}$$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & -2 & -3 \\ 0 & 0 & 6 & 6 & 0 \end{array}\right)$$

echelon form

w free
infinitely many solution

$$x + y + z + w = 10$$
$$y + 3z - 2w = -3$$
$$6z + 6w = 0$$

$$x = 10 - (5w-3) - (-w) - \cancel{w} = 13 - 5w$$
$$y = -3 - 3(-w) + 2w = 5w - 3$$
$$\Rightarrow \quad 6z = -6w \quad \Rightarrow \quad z = -w$$

$$(x,y,z,w) = (13-5w,\ 5w-3,\ -w,\ w)$$

with w free

8.

$$\left(\begin{array}{ccccc|c} \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{array}\right)$$

i) no pivot in the last col.
ii) at least two free vars.

$\Downarrow$

infinitely many solutions

9.

$$2xy + y^3 + y^2 = 0$$
$$x^2 + 3xy^2 + 2xy = 0$$

$$y(2x + y^2 + y) = 0$$
$$x(x + 3y^2 + 2y) = 0$$

$$\boxed{\begin{array}{lcl} y = 0 & \text{or} & 2x + y^2 + y = 0 \\ x = 0 & \text{"} & x + 3y^2 + 2y = 0 \end{array}}$$

a) $y=0, x=0$: $(x,y)=\underline{(0,0)}$

b) $y=0$, $x+3y^2+2y=0$: $x=0 \Rightarrow (x,y)=\underline{(0,0)}$

c) $2x+y^2+y=0$, $\underline{x=0}$: $y^2+y=0 \Rightarrow y(y+1)=0 \Rightarrow \underline{y=0}$, or $\underline{y=-1}$

$(x,y)=\underline{(0,0)}$, $\underline{\underline{(0,-1)}}$

d) $2x+y^2+y=0$, $\underbrace{x+3y^2+2y=0}_{\Updownarrow}$:

$x=\underline{-3y^2-2y}$

$2(-3y^2-2y)+y^2+y=0$

$-5y^2-3y=0$

$-y(5y+3)=0$

$\underline{y=0}$ or $y=\underline{-3/5}$

$\underline{\underline{x=0}}$ $\quad x=-3\cdot(-3/5)^2-2\cdot(-3/5)$

$= \frac{-3\cdot 9}{25}+\frac{6\cdot 5}{5\cdot 5}=\underline{\frac{3}{25}}$

$(x,y)=\underline{(0,0)}$, $=\underline{\underline{(3/25, -3/5)}}$

<u>Solutions:</u> $\underline{\underline{(0,0), (0,-1), (3/25, -3/5)}}$