Plan

Exams: final exam (5h home exam, pass/fail) counts 100% of the course this year

multiple choice: cancelled
course paper (EBA29103) will go as planned
– recommended to take it in March

① Solution of linear system:

Result: Any linear system has either
- no solutions — inconsistent
- one solution
- infinitely many solutions

} consistent

Gaussian elimination: pivot positions = the positions in the matrix of the pivots when you have an echelon form

- pivot in the last column → no solutions (inconsistent)
- no pivot in the last column ⇓ there are solutions (consistent)
  - → pivot in every variable column ⇒ one solution
  - → at least one variable column has no pivot ⇒ infinitely many solutions

{column without pivot} ⟷ {free variable}

Variable column: all columns except the last

(2) Introduction to matrices and vectors

An $m \times n$ matrix A is a rectangular array of numbers with m rows, n colums.

Ex: $A = \begin{pmatrix} 2 & 3 & 7 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ entry in A in row 1 col. 2

2×3 matrix

Matrix operations:

i) Addition / subtraction:

- position by position
- defined if the matrizes have the same size

Ex: $\begin{pmatrix} 2 & 3 & 7 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 8 \\ 3 & 2 & 4 \end{pmatrix}$

$\begin{pmatrix} 2 & 3 & 7 \\ 1 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ -1 & -4 & -4 \end{pmatrix}$

ii) Scalar multiplication:

scalar = number

Ex: $2 \cdot \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -2 & 4 \end{pmatrix}$

Square matrix: $m = n$, i.e. # rows = # cols in the matrix

An n-vector is an $n \times 1$-matrix. It is also called a column vector.

We use $\underline{v}$ as the name of a vector. $= \text{boldface } \mathbf{v} = \vec{v}$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ is a 3-vector.

Computing with vectors: addition / subtraction, scalar multiplication

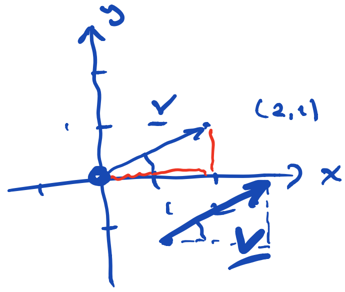
Ex: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

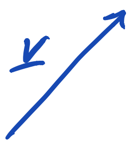
$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Geometric representation of vectors

Ex: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{v}$



A vector is a quantity that has a size and direction, represented by an arrow



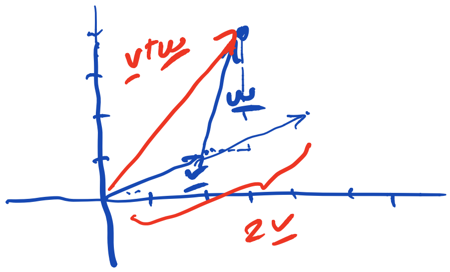
size = length of arrow
direction = direction of arrow

$||\underline{v}|| = \sqrt{2^2+1^2} = \sqrt{5}$
Length of vector $\underline{v}$

Addition of vectors

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\underline{v}$ = $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{w}$ = $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$



$2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Length of a vector:

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \longleftrightarrow \underline{v} = (v_1, v_2, \ldots, v_n)$

$||\underline{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

③ Determinants.

$A \rightsquigarrow \det(A) = |A|$

A: $n \times n$ matrix (square matrix)

determinant of A (a number)

$\underline{n=2}$:

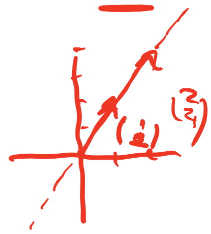
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$

Ex:

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2\cdot 3 - 0\cdot 0 = \underline{6} \qquad \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = 0 - 6 = \underline{-6}$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1\cdot 5 - 2\cdot 2 = \underline{1} \qquad \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = 2\cdot 2 - 1\cdot 5 = \underline{-1}$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 2 = \underline{0} \qquad \leftarrow \qquad \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

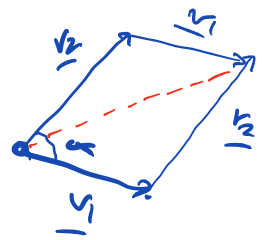


Comments:

- when we switch two rows or cols, the determinant changes sign.
- $\boxed{\text{the determinant is zero if and only if the two vectors lie on the same line} \Big\} \text{ one of the vectors is a scalar multiple of the other}}$

Geometric interpretation

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$



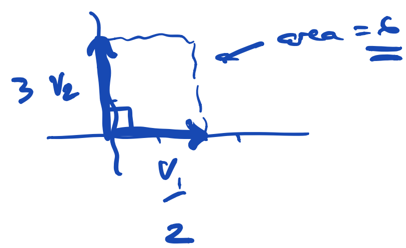
parallellogram spanned by $\underline{v}_1, \underline{v}_2$.

$\boxed{|A| = ad - bc = \pm \text{ area of parallellogram spanned by } \underline{v}_1, \underline{v}_2}$

$\| \underline{v}_1 \| \cdot \| \underline{v}_2 \| \cdot \sin\alpha$

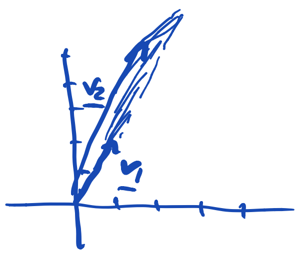
Ex: $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$|A| = 2\cdot 3 - 0 = \underline{6}$



$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$|A| = 5 - 4 = \underline{1}$



Case $n > 2$:

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \rightsquigarrow \det(A) = ?$

Method: Cofactor expansion along a row or column

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$(-1)^{1+1}$

(along first row)

$$= 1 \cdot (18-12) - (9-4) + (3-2)$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad = a_{11} \cdot \underline{C_{11}} + a_{12} \cdot \underline{C_{12}} + a_{13} \cdot \underline{C_{13}}$$

$$\begin{matrix} + & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{matrix} \qquad \begin{matrix} 1 & + & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{matrix} \qquad \begin{matrix} 1 & 1 & + \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{matrix}$$

C_{ij}: cofactor in row i, col j

Formula: $C_{ij} = (-1)^{i+j} \cdot M_{ij}$, where $\underline{M_{ij}}$ (minor) = the determinant of the matrix you get by deleting row i, col. j.

and $(-1)^{i+j}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$\oplus$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

(along 2nd col.)

$$= -(9-4) + 2 \cdot (9-1) - 3(4-1)$$

$$= -5 + 16 - 9 = \underline{\underline{2}}$$

Result:
- you can compute the determinant of any matrix using cofactor expansion
- cofactor expansion along any row or column gives the same result

$n=3$: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$ ✓

$$\boxed{|A| = +aei + bfg + cdh - ceg - afh - bdi}$$

← works for $n=3$ but not for $n>3$.

$$= a(ei - fh) + b(fg - di) + c(dh - eg)$$

$$= a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Cofactor expansion (2nd col.): $-b(di - fg) + e(ai - cg) - h(af - cd)$

$$= -\cancel{bdi} + \cancel{bfg} + \cancel{aei} - \cancel{ceg} - \cancel{afh} + \cancel{cdh}$$

Ex: $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 5 \end{vmatrix} = +1 \cdot \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 5 \end{vmatrix}$

$$= 1 \cdot \left(-1 \cdot (5 - 16) \right) - 2 \left(3 \cdot (5 - 16) \right)$$

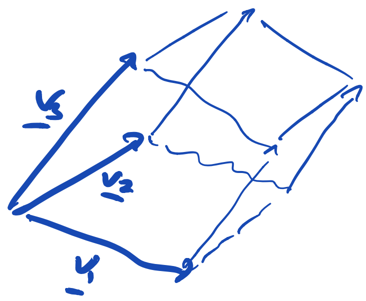
$$= -1 \cdot (-11) - 6 \cdot (-11) = 11 + 66 = \underline{\underline{77}}$$

Geometric interpretation:

3×3 determinants

$A = (\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \underline{v_3})$

$\underline{v_1}, \underline{v_2}, \underline{v_3}$: 3-vectors



$|A| = \pm$ volume of → parallellepiped

Part 2: determinants $\begin{cases} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc & n=2 \\ \text{cofactor expansion} & n>2 \end{cases}$

Problem Set 22.

4. e) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = +1 \cdot \begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & a^2 \\ 1 & b^2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}$

$= (ab^2 - a^2b) - (b^2 - a^2) + (b-a)$

$= ab(b-a) - (b-a)(b+a) + (b-a)$

$= (b-a) \cdot (ab - a - b + 1)$

$= (b-a) \cdot (a-1)(b-1) = \underline{\underline{(b-a)(a-1)(b-1)}}$

5. e) $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a \cdot \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix}$

$= a(a^2-1) - (a-1) + (1-a)$

$= a(a-1)(a+1) - (a-1) - (a-1)$

$= (a-1) \cdot (a(a+1) - 2) = (a-1)(a^2 + a - 2)$

$= (a-1)(a+2)(a-1) = \underline{(a-1)^2(a+2)}$

6a.) $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = \underline{1} \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} + \underline{1} \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$

$= 1(-1 \cdot (-1-1)) + 1(-1(1 \cdot (-1) - 1 \cdot 1))$

$= -1 \cdot (-2) - 1 \cdot (-2) = \underline{\underline{4}}$

6c,

$$\begin{vmatrix} 1 & 1 & 4 & 6 \\ 0 & 2 & \sqrt{3} & -1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot (-2) = \underline{-12}$$

upper
triangular matrix

$$= \underline{1} \cdot \begin{vmatrix} 2 & \sqrt{3} & -1 \\ 0 & 3 & 7 \\ 0 & 0 & -2 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} 3 & 7 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 3 \cdot (-2)$$

$$= \underline{-12}$$

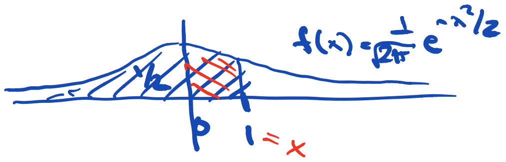
7. **Gauss(x,n):**

compute $\frac{1}{2} + \int_0^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = P(X \le x)$

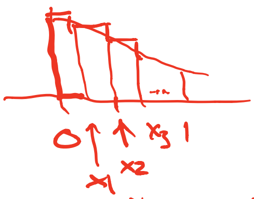
use Riemann sum with n subintervalls.

a) **Gauss(1,10):**

$n = 10$



std. normal distr.



$$\underline{\Delta x = \frac{x}{n}}$$

$$\left\{ \begin{aligned} x_0 &= 0 \\ x_1 &= 0 + \Delta x \\ x_2 &= 0 + 2 \cdot \Delta x \\ &\vdots \\ x_{n-1} &= 0 + (n-1) \cdot \Delta x \end{aligned} \right\}$$

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 0.1 \\ x_2 &= 0.2 \\ &\vdots \\ x_9 &= 0.9 \end{aligned}$$

$$0.5 + \underbrace{f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \dots + f(x_{n-1}) \cdot \Delta x}_{\text{Riemann sum}}$$

return this value

function Gauss (x,n):
    Δx = x/n
    for ... :
        xi = ...
    for ... :
        integral = integral + ...
    return (integral + 0,5)

```
import numpy as np

def Gauss(x,n):
    if (x<0):
        print("The first argument cannot be negative")
        return 0
    f = lambda x : 1/np.sqrt(2*np.pi) * np.exp(-x**2/2)
    a = 0; b = x; dx = (b-a)/n
    x_left = np.linspace(a,b-dx,n)
    return 0.5 + np.sum(f(x_left) * dx)
```