Plan

1. Determinants and linear systems
2. Linear systems with parameters
3. Vector equations

1 Determinants and linear systems

Alternative method: Determinants

$$|A| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = \underline{\underline{4}}$$

$$|A| = \left|\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\right| \; -1 \; = \; \left|\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\right| \; -1 \; = \; \left|\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}\right| = |E|$$

echelon form $\Rightarrow$ upper triangular

i) A square matrix in echelon form is upper triangular, hence the determinant is the product of diagonal entries. $|E| = 1 \cdot 1 \cdot (-2) \cdot (-2) = \underline{4}$

ii) If $A \to B$ is an elementary row operation:

a) If $A \to B$ is to switch two rows: $|B| = -|A|$

b) If $A \to B$ is to multiply a row by $c \neq 0$: $|B| = c \cdot |A|$

c) If $A \to B$ is to add a multiple of one row to another row: $|B| = |A|$

Result:

We look at an $n\times n$ linear system, and let A be the coefficient matrix of the system. Then we have:

$|A| \neq 0 \Rightarrow$ there is exactly one solution

$|A| = 0 \Rightarrow$ there are no solutions or infinitely many solutions

Ex:

$$\left.\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \\ x+3y+9z&=13 \end{aligned}\right\}$$

3×3 lin. sys.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

coeff. matrix 3×3

$|A| = 1\cdot 6 - 1\cdot 5 + 1\cdot 1 = 2 \neq 0$

$\Downarrow$

exactly one solution

Explanation of the result:

$n\times n$ linear system, coeff matrix A

$A \to \cdots \to E$ = echelon form

elementary row operations

Remark: i) $|A| = 0 \Leftrightarrow |E| = 0$

ii) $E = \begin{pmatrix} e_{11} & & * \\ & e_{22} & \\ & & \ddots \\ 0 & & & e_{nn} \end{pmatrix}$

$|E| = e_{11}\cdot e_{22}\cdot \ldots \cdot e_{nn}$

$|E| = 0 \Leftrightarrow$ at least one $e_{ii} = 0$

$\Leftrightarrow$ at least one variable col. has no pivot

$|A| \neq 0$: pivot in every variable col.

$\Rightarrow$ exactly one solution

$|A| = 0$: at least one variable col. without pivot

$\Rightarrow$ inf. many solutions or no solutions

2 Linear systems with parameters

Ex: $x + y = 4$
$x + ay = 6$

x, y variables
a parameter
$\Downarrow$
want to solve for x, y;
the solutions will depend on a

Alt I: Gauss

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & a & 6 \end{array}\right)]-1 \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right)$$

$a=1$: $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array}\right)$ no solutions

$a \neq 1$: $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right)$ one solution

$x + y = 4$
$(a-1)y = 2$

$x = 4 - y = \frac{4(a-1)}{a-1} - \frac{2}{a-1}$

$y = \frac{2}{a-1}$

$$(x, y) = \left(\frac{4a-6}{a-1}, \frac{2}{a-1}\right), \quad a \neq 1$$

Alt 2: Determinants

$A = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$ $|A| = a - 1$

$|A| = 0$: $a - 1 = 0$
$a = 1$

$\lvert A\rvert = 0$: $a = 1$	$\lvert A\rvert \neq 0$: $a \neq 1$
no solution or inf. many s.	one solution

$a=1$: $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 1 & 6 \end{array}\right)]-1 \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array}\right)$

Gauss
$\Downarrow$
no solutions

we can find formulas for x, y using Cramer's rule. $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & a & 6 \end{array}\right)$

$x = \frac{|A_x(b)|}{|A|} = \frac{\begin{vmatrix} 4 & 1 \\ 6 & a \end{vmatrix}}{a-1} = \frac{4a-6}{a-1}$

$y = \frac{|A_y(b)|}{|A|} = \frac{\begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix}}{a-1} = \frac{2}{a-1}$

Kramer's rule:

We look at a linear system that is $n \times n$, with coeff. matrix A and extended matrix $(A\,|\,\underline{b})$. If $|A| \neq 0$, then:

$$x_1 = \frac{|A_1(\underline{b})|}{|A|}, \quad x_2 = \frac{|A_2(\underline{b})|}{|A|}, \quad \dots, \quad x_n = \frac{|A_n(\underline{b})|}{|A|}$$

where $|A_i(\underline{b})|$ is the determinant you get when you replace col. #i in A with $\underline{b}$.

Ex:

$$\begin{aligned} x + y + z &= 3 \\ x + ay + 4z &= 7 \\ x + 3y + az &= 13 \end{aligned}$$

3×3 system with parameter a

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & 4 \\ 1 & 3 & a \end{pmatrix} : \quad |A| = 1 \cdot (a^2 - 12) - 1 \cdot (a - 4) + 1\,(3 - a)$$

$$= a^2 - 12 - a + 4 + 3 - a$$

$$= a^2 - 2a - 5$$

$|A| = 0$: $a^2 - 2a - 5 = 0$

$$a = \frac{2 \pm \sqrt{2^2 - 4 \cdot (-5)}}{2 \cdot 1} = \frac{2 \pm \sqrt{24}}{2}$$

$$= \frac{2 \pm \sqrt{4} \cdot \sqrt{6}}{2} = 1 \pm \sqrt{6}$$

$\lvert A\rvert = 0$:	$\lvert A\rvert \neq 0$:
$a = 1 + \sqrt{6}$: Gauss	$a \neq 1 \pm \sqrt{6}$: Kramer's rule
$a = 1 - \sqrt{6}$: Gauss	$x = \_\_ \quad y = \_\_ \quad z = \_\_$

i) $a = 1+\sqrt{6}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1+\sqrt{6} & 4 & 7 \\ 1 & 3 & 1+\sqrt{6} & 13 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & \sqrt{6} & 3 & 4 \\ 0 & 2 & \sqrt{6} & 10 \end{array}\right) \cdot \sqrt{6}/3$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & \sqrt{6} & 4\sqrt{6}/3 \\ 0 & 2 & \sqrt{6} & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & \sqrt{6} & 4\sqrt{6}/3 \\ 0 & 0 & 0 & 10-\frac{4\sqrt{6}}{3} \end{array}\right)$$

no solutions.

ii) $a = 1-\sqrt{6}$: no solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1-\sqrt{6} & 4 & 7 \\ 1 & 3 & 1-\sqrt{6} & 13 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -\sqrt{6} & 3 & 4 \\ 0 & 2 & -\sqrt{6} & 10 \end{array}\right) \cdot \frac{\sqrt{6}}{3}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 3\sqrt{6}/3 & 4\sqrt{6}/3 \\ 0 & 2 & -\sqrt{6} & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & \sqrt{6} & 4\sqrt{6}/3 \\ 0 & 0 & 0 & 10+\frac{4\sqrt{6}}{3} \end{array}\right)$$

$10 + \frac{4\sqrt{6}}{3} \neq 0$

no solutions

$a \neq 1 \pm \sqrt{6}$: Kramer's rule

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & a & 4 & 7 \\ 1 & 3 & a & 13 \end{array}\right)$$

$|A| = a^2 - 2a - 5$

$$x = \frac{|A_1(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 7 & a & 4 \\ 13 & 3 & a \end{vmatrix}}{a^2-2a-5}$$

$$= \frac{3\cdot(a^2-12) - 1\cdot(7a-52) + 1(21-13a)}{a^2-2a-5}$$

$$= \frac{3a^2 - 20a + 37}{a^2-2a-5}$$

$$y = \frac{|A_2(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 7 & 4 \\ 1 & 13 & a \end{vmatrix}}{a^2-2a-5} = \frac{(7a-52) - 3(a-4) + 6}{a^2-2a-5} = \frac{4a-34}{a^2-2a-5}$$

$$z = \frac{|A_3(b)|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & a & 7 \\ 1 & 3 & 13 \end{vmatrix}}{a^2-2a-5} = \frac{(13a-21) - (6) + 3(3-a)}{a^2-2a-5} = \frac{10a-18}{a^2-2a-5}$$

$$(x,y,z) = \left(\frac{3a^2-20a+37}{a^2-2a-5}, \frac{4a-34}{a^2-2a-5}, \frac{10a-18}{a^2-2a-5}\right)$$

for $a \neq 1 \pm \sqrt{6}$

Method:
i) Compute $|A|$ and find out when $|A| = 0$.
ii) In cases with $|A| = 0$: Gauss
iii) In cases with $|A| \neq 0$: Kramer's rule

③ Vector equations

Defn: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ n-vectors

A linear combination of $\underline{v}_1, \ldots, \underline{v}_r$ is an expression of the form

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \ldots + c_r \cdot \underline{v}_r$$

where $c_1, c_2, \ldots, c_r$ are numbers.

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

What are the linear combinations of $\underline{v}_1, \underline{v}_2, \underline{v}_3$?

vector equation.

$$\boxed{x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Question: Is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$?

$$\begin{pmatrix} x_1 \\ x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4x_3 \\ -x_3 \\ 2x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + x_2 + 4x_3 \\ x_1 + 3x_2 - x_3 \\ 2x_1 \quad + 2x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

linear system

$$\boxed{\begin{aligned} x_1 + x_2 + 4x_3 &= 1 \\ x_1 + 3x_2 - x_3 &= 0 \\ 2x_1 \quad + 2x_3 &= 0 \end{aligned}}$$

if the linear system is consistent then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a linear comb.

Gauss: $\left(\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$

Part 2:

Revision ① $n\times n$ linear system with coeff. matrix A — $|A|\neq 0$: one solution; $|A|=0$: no solutions or inf. many solutions

② In case $|A|\neq 0$: Kramer's rule

$$x_1 = \frac{|A_1(b)|}{|A|}, \dots, x_n = \frac{|A_n(b)|}{|A|}$$

Problem Set 23:

1b.
$$\left(\begin{array}{ccc|c} 2 & a & -1 & a-5 \\ -1 & 2 & a & -3 \\ a & -1 & 2 & a+10 \end{array}\right)$$

$$|A| = \begin{vmatrix} 2 & a & -1 \\ -1 & 2 & a \\ a & -1 & 2 \end{vmatrix} = 2(4+a) - a(-2-a^2) - 1(1-2a)$$
$$= 8 + 2a + 2a + a^3 - 1 + 2a$$
$$= a^3 + 6a + 7$$

$|A|=0$: $a^3+6a+7=0$ $\quad \pm1, \pm7 \quad (-1)^3+6\cdot(-1)+7=0 \ \checkmark \ x=-1$

$(a+1)(a^2-a+7)=0$

$a=-1$ or $a^2-a+7=0$

$a = \frac{1\pm\sqrt{1-4\cdot 7}}{2}$ no solutions

$a\neq -1$: one solution

$a=-1$:
$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & -6 \\ -1 & 2 & -1 & -3 \\ -1 & -1 & 2 & 9 \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -9 \\ -1 & 2 & -1 & -3 \\ -1 & -1 & 2 & 9 \end{array}\right)$$

$$\xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -9 \\ 0 & 3 & -3 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

inf. many solutions (1 free)

4b. $\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 2 \\ 1 & 1 & a & -3 \end{array}\right)$

$$|A| = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a(a^2-1) - 1\cdot(a-1) + 1\cdot(1-a)$$

$$= a(a-1)(a+1) - 1\cdot(a-1) - 1\cdot(a-1)$$

$$= (a-1)\cdot[a(a+1) - 1 - 1]$$

$$= (a-1)(a^2+a-2)$$

$$= (a-1)(a+2)(a-1) = (a-1)^2(a+2)$$

$|A|=0$: $a=1$, $a=-2$

$a=1$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -3 \end{array}\right)$ no solutions

$a=-2$: $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & -2 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & -1 & 2 & 3 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & -2 & -3 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} -1 & -1 & 2 & 3 \\ 0 & -3 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$ infinitely many solutions (z free)

$$-x - y + 2z = 3$$
$$-3y + 3z = 5$$

$$\frac{-3y}{-3} = \frac{-3z+5}{-3} = z - 5/3$$

$$-x = 3 + y - 2z$$
$$= 3 + z - 5/3 - 2z$$
$$= -z + 4/3$$
$$x = z - 4/3$$

$(x,y,z) = (z - 4/3,\ z - 5/3,\ z)$ with z free when $a=-2$

$a \neq 1, -2$: $|A| \neq 0 \Rightarrow$ one solution

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 2 \\ 1 & 1 & a & -3 \end{array}\right) \qquad |A| = (a-1)^2(a+2)$$

Kramer's rule:

$$x = \frac{|A_1(b)|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & a & 1 \\ -3 & 1 & a \end{vmatrix}}{(a-1)^2(a+2)} = \frac{(a^2-1) - (2a+3) + (2+3a)}{*}$$

$$= \frac{a^2+a-2}{(a-1)^2(a+2)} = \frac{\cancel{(a+2)}\cancel{(a-1)}}{(a-1)^{\cancel{2}}\cancel{(a+2)}} = \frac{1}{a-1} \quad ✓$$

$$y = \frac{|A_2(b)|}{|A|} = \frac{\begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -3 & a \end{vmatrix}}{|A|} = \frac{a(2a+3) - (a-1) + (-5)}{|A|}$$

$$= \frac{2a^2+2a-4}{(a-1)^2(a+2)} = \frac{2\cancel{(a+2)}\cancel{(a-1)}}{(a-1)^{\cancel{2}}\cancel{(a+2)}} = \frac{2}{a-1} \quad ✓$$

$$z = \frac{|A_3(b)|}{|A|} = \frac{\begin{vmatrix} a & 1 & 1 \\ 1 & a & 2 \\ 1 & 1 & -3 \end{vmatrix}}{|A|} = \frac{a(-3a-2) - (-5) + (1-a)}{|A|}$$

$$= \frac{-3a^2-3a+6}{(a-1)^2(a+2)} = \frac{-3\cancel{(a+2)}\cancel{(a-1)}}{(a-1)^{\cancel{2}}\cancel{(a+2)}} = \frac{-3}{a-1} \quad ✓$$

$$(x,y,z) = \left(\frac{1}{a-1}, \frac{2}{a-1}, \frac{-3}{a-1}\right), \quad a \neq 1, -2$$

8. $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 7 \end{pmatrix}$ Find all linear combinations on $\underline{v}_1, \underline{v}_2, \underline{v}_3$.

$\underline{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$: Find all (a,b,c,d) such that $\underline{b}$ is a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$.

$$x_1 \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3 = \underline{b} : \quad x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Linear system: with parameters a,b,c,d

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & -1 & b \\ 1 & 4 & 1 & c \\ 1 & 3 & 7 & d \end{array}\right) \begin{matrix}]\cdot(-1) \\ \\ \\ \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 3 & 0 & c-a \\ 0 & 2 & 6 & d-a \end{array}\right) \begin{matrix} \\ ]\cdot(-3) \\ ]\cdot(-2) \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 6 & \boxed{c-a-3(b-a)} \\ 0 & 0 & 10 & \boxed{d-a-2(b-a)} \end{array}\right) \begin{matrix} \\ \\ ]\cdot\left(-\frac{10}{6}\right) \\ \end{matrix}$$

$c-3b+2a$

$d-2b+a$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 6 & c-3b+2a \\ 0 & 0 & 0 & * \end{array}\right)$$

$* = 0$ ⇓ one solution **Yes**

$* \neq 0$ ⇓ no solution **No**

$$* = \frac{6}{6}(d-2b+a) - \frac{10}{6}(c-3b+2a)$$

$$= \frac{6d-12b+6a-10c+30b-20a}{6}$$

$$* = \frac{1}{6}(-14a+18b-10c+6d)$$

$\underline{b}$ is lin. comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

$\Updownarrow$

$$\underline{\underline{-14a+18b-10c+6d=0}}$$

$\underline{(0,0,1,1)}$: $-14\cdot 0 + 18 \cdot 0 - 10 \cdot 1 + 6 \cdot 1 = -4 \neq 0$

$\Rightarrow$ $\underline{\text{not}}$ a lin. comb.

$$\begin{vmatrix} 1 & 1 & 1 & a \\ 1 & 2 & -1 & b \\ 1 & 4 & 1 & c \\ 1 & 3 & 7 & d \end{vmatrix} \rightarrow \ldots \rightarrow \begin{vmatrix} 1 & 1 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 6 & c-3b+2a \\ 0 & 0 & 0 & * \end{vmatrix} = 1\cdot 1\cdot 6\cdot *$$

$$= \cancel{6}\cdot \frac{-14a+18b-10c+6d}{\cancel{6}}$$

$$= \underline{-14a+18b-10c+6d}$$