Plan

1. Matrix multiplication
2. The transpose matrix
3. Inverse matrices

① Matrix multiplication

Addition / subtraction: $A+B$, $A-B$

Scalar multiplication: $r \cdot A$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 0 \end{pmatrix}$

$-2 \cdot \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -2 & -2 \end{pmatrix}$

Matrix multiplication: $\underset{m \times n}{A} \cdot \underset{n \times p}{B} \rightsquigarrow \underset{m \times p}{AB}$

* AB is defined if # cols in A = # rows in B.
* If so, AB has # rows = # rows in A, # cols = # cols in B
* $AB = (c_{ij})$ with $c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$

Ex: $\underset{2\times 2}{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}} \cdot \underset{2\times 1}{\begin{pmatrix} 5 \\ 3 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 10 \\ 8 \end{pmatrix}}$

$2 \cdot 5 + 0 \cdot 3 = 10$
$1 \cdot 5 + 1 \cdot 3 = 8$

Ex: $\underset{2\times 2}{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}} \cdot \underset{2\times 2}{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}}$

Ex:

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 1 & 5 & 6 \\ 2 & 5 & 7 \end{pmatrix}}$$

$$\begin{array}{c|c} & \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 5 & 6 \\ 2 & 5 & 7 \end{pmatrix} \end{array}$$

Remarks:

i) $AB \neq BA$!

ii) If A is square ($n\times n$), we can compute powers of A:

$$A\cdot A = A^2$$
$$(A\cdot A)A = A^3$$
$$\vdots$$

A^m is defined when A is square and m is positive integer.

Linear systems and matrix multiplication.

$$m \text{ eqns} \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n}_{n \text{ variables}} = b_m \end{array} \right\} \Rightarrow$$

Matrix form of lin. sys:

$A\cdot \underline{x} = \underline{b}$

$(A \mid \underline{b})$ extended matrix $\quad A$

$$A = \underset{\substack{\text{coeff. matrix} \\ m\times n}}{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}} \qquad \underline{x} = \underset{n\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} \qquad \underline{b} = \underset{m\times 1}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}$$

$$A\cdot \underline{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$= \underline{b}$$

2 Inverse matrices

Definition: An identity matrix is a square matrix with 1's on the diagonal and 0's everywhere else. It is written I.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Property: $\left.\begin{aligned} A \cdot I = A \\ I \cdot A = A \end{aligned}\right\}$ for all matrices A.

Ex: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\underset{2\times 3}{\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \cdot \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Definition: Let A be any matrix. An inverse A^{-1} of A is a matrix such that $\left\{\begin{aligned} A^{-1} \cdot A = I \\ A \cdot A^{-1} = I \end{aligned}\right\}$

Numbers:

① $2x = 6$

$\cancel{2^{-1}} \cdot \cancel{2}x = \frac{1}{2} \cdot 6$

$2^{-1} = \frac{1}{2}$

$x = 3$

$\frac{1}{2} \cdot 2 = 1$

$2^{-1} \cdot 2 = 1$

Facts:

i) If A has an inverse, then it is unique.

ii) A has an inverse if and only if A is square and $|A| \neq 0$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ has an inverse

$|A| = 1 - 2 = -1 \neq 0$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has no inverse

$|A| = 4 - 4 = 0$

i) The case $n=2$:

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad A^{-1}=\frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad-bc\neq 0$$

A^{-1} does not exist, $ad-bc=0$

Ex: $A=\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ $\quad A^{-1}=\frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}=\begin{pmatrix} 3/2 & -1/2 \\ -2 & 1 \end{pmatrix}$

$|A|=2\cdot 3-4\cdot 1=2\neq 0$

$$\underbrace{\boxed{\frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}}}_{A^{-1}}\cdot\underbrace{\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}}_{A}=\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$$

$$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex:
$$2x+y=8$$
$$4x+3y=18$$

Matrix form: $A\cdot\underline{x}=\underline{b}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 8 \\ 18 \end{pmatrix}$$

$A\underline{x}=\underline{b} \quad | A^{-1}\cdot$

$\cancel{A^{-1}\cdot A}\underline{x}=A^{-1}\cdot\underline{b}$

$I\underline{x}=A^{-1}\cdot\underline{b}$

$\boxed{\underline{x}=A^{-1}\cdot\underline{b}}$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\underbrace{\frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}}_{A^{-1}}\cdot\underbrace{\begin{pmatrix} 8 \\ 18 \end{pmatrix}}_{\underline{b}}$$

$$=\frac{1}{2}\cdot\begin{pmatrix} 6 \\ 4 \end{pmatrix}=\underline{\underline{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}}$$

ii) The case $n > 2$:

A ($n \times n$ matrix):
- $A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$, $|A| \neq 0$
- A^{-1} does not exist when $|A| = 0$

Defn: $\text{adj}(A)$ = the adjungated matrix of A
$= C^T$, where C is the cofactor matrix of A, and C^T is the transpose of A.

$$\underset{n\times n}{A} \rightsquigarrow C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \qquad C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

cofactor matrix

$$\underset{m\times n}{A} \rightsquigarrow \underset{n\times m}{A^T}$$

the transpose of A (switch rows and columns)

Ex: $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\boxed{(AB)^T = B^T \cdot A^T}$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$|A| = +1 \cdot (18-12) - 1 \cdot (9-4) + 1(3-2)$
$= 2 \neq 0$

$$A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$= \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & +1 \end{pmatrix}^T = \underline{\underline{\frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

Problem Set 23:

9.

	A	B	C
	60	75	320
1	80	80	350
2	100	25	500
3	40	100	55

	A	B	C
1	20	5	30
2	40	-50	180
3	-20	25	-265

Returns

$C = 400.000$:

$$\begin{pmatrix} 80 & 80 & 350 \\ 100 & 25 & 500 \\ 40 & 100 & 55 \end{pmatrix} - \begin{pmatrix} 60 & 75 & 320 \\ 60 & 75 & 320 \\ 60 & 75 & 320 \end{pmatrix} =$$

$$20x + 5y + 30z = R_1$$
$$40x - 50y + 180z = R_2$$
$$-20x + 25y - 265z = R_3$$
$$\boxed{60x + 75y + 320z = C}$$

b) $R_1 = 50'$, $R_2 = 25'$, $R_3 = -100'$, $C = 400'$

$$\left(\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 40 & -50 & 180 & R_2 \\ -20 & 25 & -265 & R_3 \\ 60 & 75 & 320 & C \end{array}\right) \to \dots \to \left(\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + \frac{1}{2}R_2 \\ 0 & 0 & 0 & * \end{array}\right)$$

-2, $+1$, -3

$$* = C - 5R_1 + 2R_2 + 2R_3 = 0$$

$* \neq 0$: no solution

$* = 0$: One solution (x, y, z) $\Leftrightarrow$ $\boxed{5R_1 - 2R_2 - 2R_3 = 400'}$

$R_1 = 50'$, $R_2 = 25'$, $R_3 = -100'$: ok

c) $R_1 > 0$, $R_2 = R_3 = 0$: $5R_1 - 2 \cdot 0 - 2 \cdot 0 = 400.000$

$R_1 = 80.000$, $R_2 = R_3 = 0$

d) For ex: $R_1 = R_2 = R_3$

$5R_1 - 2R_1 - 2R_1 = 400'$ $\Rightarrow$ $R_1 = R_2 = R_3 = 400.000$

Part 2: Review: - Matrix multiplication
- Inverse matrices

A^{-1} exist $\Leftrightarrow$ $|A| \neq 0$

$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$

Problem set 24:

2f) $A = \begin{pmatrix} 7 & 1 & 4 \\ -2 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix}$

$|A| = 3 \cdot (-6) - 3(-14+8)$

$= -18 + 18 = 0$

A^{-1} does not exist

3c) $A = \begin{pmatrix} 1 & 1 & a \\ 1 & 3 & 1 \\ a & 1 & 1 \end{pmatrix}$

$|A| = 1 \cdot (2) - 1 \cdot (1-a) + a(1-3a)$

$= 2 - 1 + a + a - 3a^2$

$= -3a^2 + 2a + 1$

$|A| = 0$: $-3a^2 + 2a + 1 = 0$

$a = \frac{-2 \pm \sqrt{2^2 - 4(-3) \cdot 1}}{2 \cdot (-3)}$

$a = \frac{-2 \pm 4}{-6} = 1, -1/3$

A^{-1} exists when $a \neq 1, -1/3$:

$A^{-1} = \frac{1}{-3a^2 + 2a + 1} \cdot \text{adj}(A)$

$C_{11} = +2$ $C_{12} = -(1-a)$ $C_{13} = 1-3a$

$C_{21} = -(1-a)$ $C_{22} = 1-a^2$ $C_{23} = -(1-a)$

$C_{31} = 1-3a$ $C_{32} = -(1-a)$ $C_{33} = 2$

cofactor matrix C

$= \frac{1}{1+2a-3a^2} \cdot \begin{pmatrix} 2 & a-1 & 1-3a \\ a-1 & 1-a^2 & a-1 \\ 1-3a & a-1 & 2 \end{pmatrix}^T$

$= \frac{1}{1+2a-3a^2} \begin{pmatrix} 2 & a-1 & 1-3a \\ a-1 & 1-a^2 & a-1 \\ 1-3a & a-1 & 2 \end{pmatrix}$

5. c) $A(3B-C)+(A-2B)C+2B(C+2A)$

$$= 3AB - \cancel{AC} + \cancel{AC} - \cancel{2BC} + \cancel{2BC} + 4BA$$

$$= \underline{3AB + 4BA}$$

e) $(BAB^{-1})^2 \cdot B^2 = (BA\cancel{B^{-1}})(\cancel{B}AB^{-1})\cdot \cancel{B}\cdot B$

$$= BA\cdot A\cdot B = \underline{BA^2B}$$

$$-(C-B)(C-A) \qquad\qquad (C-A)(C-A)$$

f) $(A-B)(C-A) + (C-B)(A-C) + (C-A)^2$

$$= (\cancel{A}-\cancel{B} - (\cancel{C}-\cancel{B}) + \cancel{C}-\cancel{A})(C-A) = (0)(C-A) = \underline{0}$$

Alt: $\cancel{AC} - \cancel{BC} - \cancel{A^2} + \cancel{BA} + \cancel{CA} - \cancel{BA} - \cancel{C^2} + \cancel{BC} + \cancel{C^2} - \cancel{AC} - \cancel{CA} + \cancel{A^2}$

$$= 0$$

1. i) $BA = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & 3 & 6 \\ 2 & 0 & 0 \\ 1 & 7 & 13 \end{pmatrix}}$

4. $A\underline{x} = b$ where $A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$

a) Gauss: $t=2$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

One solution

$3z = 2 \qquad z = \underline{2/3}$

$2y = 0 \qquad y = \underline{0}$

$x - z = 0 \qquad x = z = \underline{2/3}$

$(x,y,z) = \underline{(2/3, 0, 2/3)}$

b) $A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ $|A| = +t(t^2-1) = t(t-1)(t+1)$

$|A|=0$: $t=0$, $t=1$, $t=-1$

(x) $t \neq 0,1,-1$: $|A| \neq 0 \Rightarrow$ one solution

$t=0$: $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ inf. many solutions

$t=1$: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ inf. many solutions

$t=-1$: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{1} \left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$ no solutions

c) $A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ $|A| = t(t-1)(t+1)$

$|A|=0$: $t=0$, $t=1$, $t=-1$

A^{-1} exists for $t \neq 0,1,-1$: $|A| \neq 0$

$$A^{-1} = \frac{1}{t(t-1)(t+1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}^T = \frac{1}{t(t-1)(t+1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}$$

Solve $A\underline{x}=\underline{b}$ when $t \neq 0,1,-1$:

$A\underline{x} = \underline{b}$

$\cancel{A^{-1}A}\underline{x} = A^{-1}\cdot\underline{b}$

$\underline{x} = A^{-1}\cdot\underline{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\frac{1}{t(t-1)(t+1)} \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} t \\ 0 \\ t \end{pmatrix}}_{\underline{b}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{t(t-1)(t+1)} \cdot \begin{pmatrix} t^3 - t^2 \\ 0 \\ -t^2 + t^3 \end{pmatrix} = \frac{1}{t(t-1)(t+2)} \cdot \begin{pmatrix} t^2(t-1) \\ 0 \\ t^2(t-1) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t^{2}\cancel{(t-1)}}{\cancel{t}\cancel{(t-1)}(t+1)} \\ 0 \\ t/(t+1) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t/(t+1) \\ 0 \\ t/(t+1) \end{pmatrix}, \quad t \neq 0, 1, -1$$