Plan

1 Matrix algebra
2 Exam MET11803 05/2018 Question 3

① Matrix algebra

a) Addition, subtraction, multiplication

Note: $A \cdot B \neq B \cdot A$

Most other algebra rules hold for matrices

Ex: $A \cdot (B+C) = A \cdot B + A \cdot C$ ✓

$A + B = B + A$

$A \cdot (B \cdot C) = (A \cdot B) \cdot C$

$(A+B)^2 = (A+B) \cdot (A+B)$

$= A \cdot A + B \cdot A + AB + B \cdot B$

$= A^2 + BA + AB + B^2$

b) Determinants:

i) $|A \cdot B| = |A| \cdot |B|$ ✓

ii) $|c \cdot A| = c^n \cdot |A|$ if A is $n \times n$, c scalar

iii) $|A^T| = |A|$

iv) $|A^{-1}| = \frac{1}{|A|}$

$$\left\{ \begin{array}{l} A \cdot A^{-1} = I \\ |A \cdot A^{-1}| = |I| \\ |A| \cdot |A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|} \end{array} \right.$$

$$\left.\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ B &= \begin{pmatrix} u & v \\ w & r \end{pmatrix} \end{aligned}\right\}$$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u & v \\ w & r \end{pmatrix} = \begin{pmatrix} au+bw & av+br \\ cu+dw & cv+dr \end{pmatrix}$$

$$|AB| = (au+bw)\cdot(cv+dr) - (cu+dw)(av+br)$$

$$\shortparallel$$

$$|A|\cdot|B| = (ad-bc)\cdot(ur-vw)$$

$$\boxed{|AB| = |A|\cdot|B|}$$

c) <u>Transpose</u> : $\underset{m\times n}{A} \rightsquigarrow \underset{n\times m}{A^T}$

<u>Ex:</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ A's <u>not</u> symmetric

i) $(AB)^T = B^T \cdot A^T$ ✓

ii) <u>Defn:</u> A is called <u>symmetric</u> if $A^T = A$.

<u>Ex:</u> $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 7 \end{pmatrix}$ is <u>symmetric</u>

d) <u>Inverses</u>

i) $(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}$ ✓

$$(AB)\cdot(B^{-1}A^{-1}) = I$$
$$\shortparallel$$
$$A\cancel{B}\cancel{B^{-1}}A^{-1}$$
$$\shortparallel$$
$$AA^{-1}$$
$$\shortparallel$$
$$I$$

(2) Exam MET11803 05/2018, Q3

$A\underline{x} = \underline{b}$ with $A = \begin{pmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 3 \\ a \\ -a \end{pmatrix}$

where a is a parameter

a) Solve $A\underline{x} = \underline{b}$ when $\underline{a=1}$.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form
z free

$\underline{x} \quad -z = 2 \qquad x = \underline{z+2}$

$\underline{y} + 2z = -1 \qquad y = \underline{-2z-1}$

Solution: $(x, y, z) = \underline{(z+2, -2z-1, z)}$ with z free

b) Find $|A|$, and determine when $|A| = 0$.

$$|A| = \begin{vmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{vmatrix} = 2(2-a^2) - a(2a-0) + 0$$

$$= 4 - 2a^2 - 2a^2 = \underline{4 - 4a^2}$$

$$= 4(1-a^2) = \underline{4(1-a)(1+a)}$$

$\underline{|A| = 0}$: $4(1-a)(1+a) = 0$

$\underline{a=1}$ or $\underline{a=-1}$

$\underline{|A| = 0 \text{ when } a=1 \text{ or } a=-1}$

c) Find all values of such that $A\underline{x}=\underline{b}$ has infinitely many solutions.

$|A| \neq 0 \implies$ there is exactly one solution

$|A| = 0 \implies$ no solution or inf. many solution

$\underline{|A|=0}$: $a=1, a=-1$ from (b).

$\underline{a=1}$: Inf. many solutions from (a).

$\underline{a=-1}$: $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{array}\right)]_1 \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{array}\right)]_1$

$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & 1 \end{array}\right)]_1 \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$

echelon form

no solutions

The system has inf. many solutions for $a=1$.

d) Compute $\underline{x}^T A \underline{x}$ when $a=1$. $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underline{x}^T = (x\ y\ z)$

$$\underline{x}^T \cdot A \cdot \underline{x} = (x\ y\ z) \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (2x+y \quad x+y+z \quad y+2z) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left((2x+y)x + (x+y+z)y + (y+2z)z\right)$$

$$= (2x^2+xy+xy+y^2+yz+yz+2z^2) = \underline{\underline{(2x^2+2xy+y^2+2yz+2z^2)}}$$

e) Find the solution when $a \neq \pm 1$.

$$A = \begin{pmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ a \\ -a \end{pmatrix} \qquad |A| = 4(1-a)(1+a) \neq 0$$

for $a \neq \pm 1$.

Methods: i) Kramer's rule

ii) Use A^{-1}: $A\underline{x} = \underline{b}$

$A^{-1}A\underline{x} = A^{-1}\underline{b}$ $\quad \underline{x} = A^{-1} \cdot \underline{b}$

iii) Gauss?

i) Kramers rule

$$x = \frac{|A_x(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 3 & a & 0 \\ a & 1 & a \\ -a & a & 2 \end{vmatrix}}{4(1-a)(1+a)} = \frac{3(2-a^2) - a(2a+a^2)}{4(1-a)(1+a)}$$

$$= \frac{-a^3 - 5a^2 + 6}{4(1-a)(1+a)}$$

$$y = \frac{|A_y(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 2 & 3 & 0 \\ a & a & a \\ 0 & -a & 2 \end{vmatrix}}{4(1-a)(1+a)} = \frac{2(2a+a^2) - 3(2a)}{4(1-a)(1+a)}$$

$$= \frac{2a^2 - 2a}{4(1-a)(1+a)} = \frac{\cancel{2}a\,(\cancel{a-1})^{-1}}{2\cancel{4}\cancel{(1-a)}(1+a)} = \frac{-a}{2(1+a)}$$

$$z = \frac{|A_z(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 2 & a & 3 \\ a & 1 & a \\ 0 & a & -a \end{vmatrix}}{4(1-a)(1+a)} = \frac{2(-a-a^2) - a(-a^2-3a)}{4(1-a)(1+a)}$$

$$= \frac{a^3 + a^2 - 2a}{4(1-a)(1+a)} = \frac{a^3 + a^2 - 2a}{4(1-a)(1+a)} \leftarrow$$

$$A = \begin{pmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{pmatrix}$$

ii) Use A^{-1}:

$$A^{-1} = \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 2-a^2 & -2a & a^2 \\ -2a & 4 & -2a \\ a^2 & -2a & 2-a^2 \end{pmatrix}^T$$

note:

A is symmetric

C is symmetric

$$= \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 2-a^2 & -2a & a^2 \\ -2a & 4 & -2a \\ a^2 & -2a & 2-a^2 \end{pmatrix}$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1} \cdot \underline{b} = \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 2-a^2 & -2a & a^2 \\ -2a & 4 & -2a \\ a^2 & -2a & 2-a^2 \end{pmatrix} \begin{pmatrix} 3 \\ a \\ -a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 3(2-a^2) - 2a \cdot a + a^2(-a) \\ -6a + 4a + 2a^2 \\ 3a^2 - 2a^2 - a(2-a^2) \end{pmatrix}$$

$$= \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 6 - 5a^2 - a^3 \\ 2a^2 - 2a \\ a^3 + a^2 - 2a \end{pmatrix}$$

(iii) Gauss: $a \neq \pm 1$

$$\left(\begin{array}{ccc|c} 2 & a & 0 & 3 \\ a & 1 & a & a \\ 0 & a & 2 & -a \end{array}\right) -a/2 \to \left(\begin{array}{ccc|c} 2 & a & 0 & 3 \\ 0 & 1-a^2/2 & a & a - 3a/2 \\ 0 & a & 2 & -a \end{array}\right) \cdot 2$$

$$\to \left(\begin{array}{ccc|c} 2 & a & 0 & 3 \\ 0 & 2-a^2 & 2a & -a \\ 0 & a & 2 & -a \end{array}\right) a \to \left(\begin{array}{ccc|c} 2 & a & 0 & 3 \\ 0 & 2 & 4a & -a-a^2 \\ 0 & a & 2 & -a \end{array}\right) -a/2$$

$$\to \left(\begin{array}{ccc|c} 2 & a & 0 & 3 \\ 0 & 2 & 4a & -a-a^2 \\ 0 & 0 & 2-2a^2 & -a+a^2/2+a^3/2 \end{array}\right) \begin{array}{l} \\ \leftarrow y \\ \leftarrow z \end{array}$$

one solution

$$(2-2a^2)z = -a + a^2/2 + a^3/2 \quad | \cdot 2$$

$$\frac{(4-4a^2)z}{4-4a^2} = \frac{-2a + a^2 + a^3}{4-4a^2}$$

$$z = \frac{a^3 + a^2 - 2a}{4-4a^2}$$

You can continue and find y and x by back substitution, but the other methods are better.

Part 2:

4. $A = \begin{pmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{pmatrix}$ $\quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 3 \\ s+4 \\ 1-2s \end{pmatrix}$

$A\underline{x} = \underline{b}$:

a) $\underline{s=8}$:

$$\left(\begin{array}{ccc|c} -6 & 3 & 3 & 3 \\ 3 & -6 & 3 & 12 \\ 3 & 3 & -6 & -15 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -3 & -3 & 6 & 15 \\ 3 & -6 & 3 & 12 \\ 3 & 3 & -6 & -15 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} -3 & -3 & 6 & 15 \\ 0 & -9 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z free $\Rightarrow$ <u>one</u> degree of freedom (one free var.)

$$-3\underline{x} - 3y + 6z = 15$$
$$-9\underline{y} + 9z = 27 \qquad -9y = -9z + 27$$
$$y = \underline{z - 3}$$

$$-3x = 3y - 6z + 15 = 3(z-3) - 6z + 15$$
$$\underline{-3x} = \frac{-3z + 6}{-3} \qquad x = \underline{z - 2}$$

$$(x, y, z) = \underline{\underline{(z-2, z-3, z)}} \quad \text{with } z \text{ free}$$

b) $\begin{vmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{vmatrix} = (2-s)\cdot\left[(2-s)^2 - 9\right]$
$$-3\left(3(2-s) - 9\right)$$
$$+3\left(9 - 3(2-s)\right)$$

$$= (2-s)(s^2-4s-5) - 3(-3-3s) + 3(3+3s)$$

$$= (2-s)(s-5)(s+1) + 18(s+1) \longleftarrow 18s+18$$

$$= (s+1)\big((2-s)(s-5)+18\big) = (s+1)(-s^2+7s+8)$$

$$= -(s+1)(s^2-7s-8) = -(s+1)^2(s-8)$$

c) $s=0$: $A = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$ $|A| = 8$

$$A^{-1} = \frac{1}{8}\begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}^T = \frac{1}{8}\begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\cdot \underline{b}$$

$$= \frac{1}{8}\begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 \\ -8 \\ 16 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

d) Exactly one solution: $|A| \neq 0$

$$\left.\begin{aligned} |A| = -(s+1)^2(s-8) = 0 \\ s=-1,\ s=8 \end{aligned}\right\}$$ exactly one solution for $s \neq -1, 8$

Kramer's rule:

$$x = \frac{|A_x(b)|}{|A|} = \frac{\begin{vmatrix} 3 & 3 & 3 \\ s+4 & 2-s & 3 \\ 1-2s & 3 & 2-s \end{vmatrix}}{-(s+1)^2(s-8)} = \frac{0}{-(s+1)^2(s-8)} = 0$$

$$\begin{vmatrix} 3 & 3 & 3 \\ s+4 & 2-s & 3 \\ 1-2s & 3 & 2-s \end{vmatrix} = 3\left((2-s)^2-9\right) - 3\left((s+4)(2-s)-3(1-2s)\right) + 3\left(3(s+4)-(2-s)(1-2s)\right)$$

$$= 3(s^2-4s-5) - 3(-s^2-2s+8-3+6s) + 3(3s+12-2s^2+5s-2)$$

$$= \cancel{3s^2} \cancel{-12s} \cancel{-15} + \cancel{3s^2} \cancel{-12s} \cancel{-15} \cancel{-6s^2} + \cancel{24s} + \cancel{30} = \underline{0}$$

3d. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$ when $a=1$

$$A^2 - 3A = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}} - 3 \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 & 4 \\ 6 & 14 & 7 \\ 4 & 7 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 3 \\ 3 & 6 & 9 \\ 3 & 9 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 1 \\ 3 & 8 & -2 \\ 1 & -2 & 10 \end{pmatrix}$$

Eivind Eriksen, Office B4-032, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/EBA2910/
BI Norwegian Business School
BSc in Business Analytics