Plan

1 Functions in two variables
2 Graph and level curves
3 Linear functions
4 Partial derivatives

1 Functions in two variables

Ex: $f(x,y) = x+3y$, $f(x,y) = x^2+y^2$ } polynomials

$f(x,y) = \dfrac{x+y}{x-y}$ } rational function

In general: $z = f(x,y)$

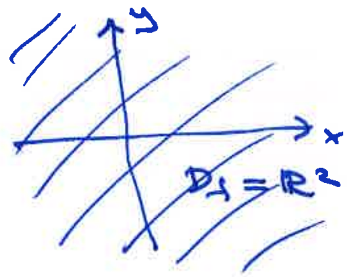
$f(x,y) = \underbrace{\text{expr. in } x \text{ and } y}_{\text{functional expression}}$

Defn: The domain of definition D_f of a function f is the set of all (x,y) we can use as inputs in f.

Ex: $f(x,y) = x^2+y^2$

$D_f = \mathbb{R}^2$

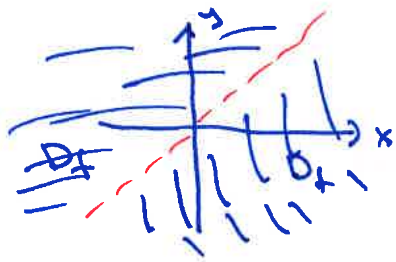
= all pairs (x,y) in two-dim. coordinate system



Ex: $f(x,y) = \dfrac{x+y}{x-y}$

D_f: $x-y \neq 0$

$D_f = \{(x,y) : x-y \neq 0\}$



The range of a function f, which is written V_f, are all values $z = f(x,y)$ with (x,y) in D_f.

Ex: $f(x,y) = x+3y$ $\quad V_f = \mathbb{R} = (-\infty,\infty)$

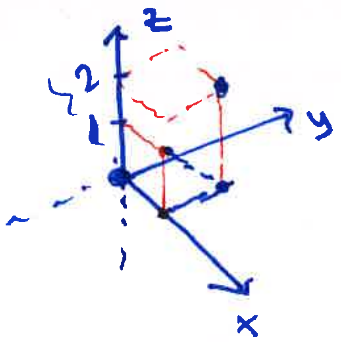
Ex: $f(x,y) = x^2+y^2$ $\quad V_f = [0,\infty)$

Ex: $f(x,y) = \dfrac{x+y}{x-y}$

In general, it is not so easy to see what V_f is for more complicated fu's; we must find max/min.

② Graph of f and level curves.

Defn: The graph of a function f in two variables is the set of pts (x,y,z) such that (x,y) in D_f and $z = f(x,y)$. $\longleftarrow$ $z = f(x,y)$



Ex: $f(x,y) = x^2+y^2$

(x,y)	$z=f(x,y)=x^2+y^2$	
$(0,0)$	$0^2+0^2=0$	$(0,0,0)$
$(1,0)$	$1^2+0^2=1$	$(1,0,1)$
$(1,1)$	$1^2+1^2=2$	$(1,1,2)$

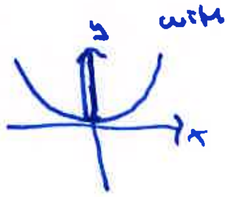
The case of a fu. in one variable:

$y = f(x)$: D_f = all values of x that we can use in f

V_f = all values $y = f(x)$ we can get from f

Graph of f:
All pts (x,y) s.t. x is in D_f and $y = f(x)$.

Ex: $f(x) = x^2$
$D_f = \mathbb{R}$
$V_f = [0,\infty)$
Graph: (x,y) with $y = x^2$



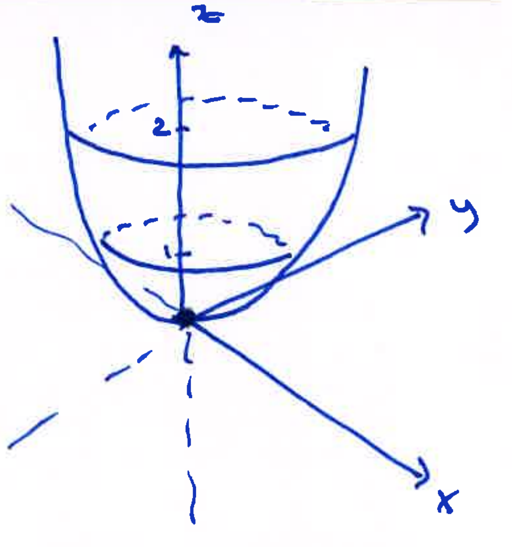
Level curves:

Defn: The level curve of f in height c is the curve

$$f(x,y) = c$$

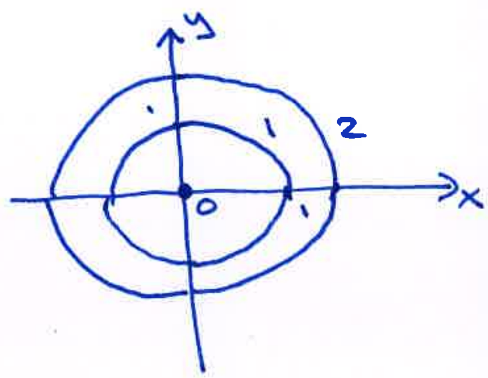
where c is a constant

→ All pts on the graph of f in height $z=c$.



Graph of $f(x,y)=x^2+y^2$

Ex: $f(x,y) = x^2+y^2$

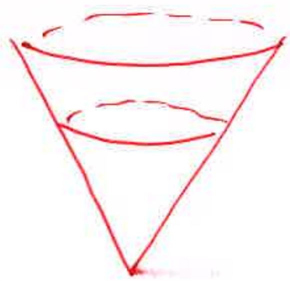


$c=1$: $f(x,y)=1$ circle w/center $(0,0)$
$x^2+y^2=1$ and $r=\sqrt{1}=1$

$c=2$: $f(x,y)=2$ — " —
$x^2+y^2=2$ with $r=\sqrt{2}$

$c=0$: $f(x,y)=0$ pt. $(0,0)$
$x^2+y^2=0$

$c<0$: $x^2+y^2=c$ no points



Cuts: $z=c$ → level curves

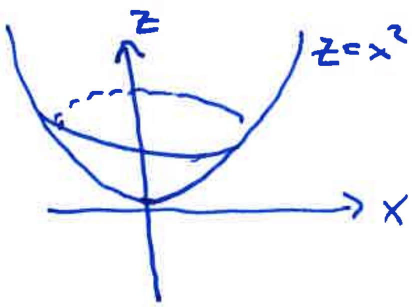
$\left.\begin{matrix} x=a \\ \text{or} \\ y=b \end{matrix}\right\}$ → vertical cuts

Ex: $f(x,y) = x^2+y^2$

Cut $y=0$:

$z = f(x,0) = x^2+0^2$

$z = x^2$



Ex: $f(x,y) = x^2 - y^2$

What are the level curves

— " — the vertical cuts

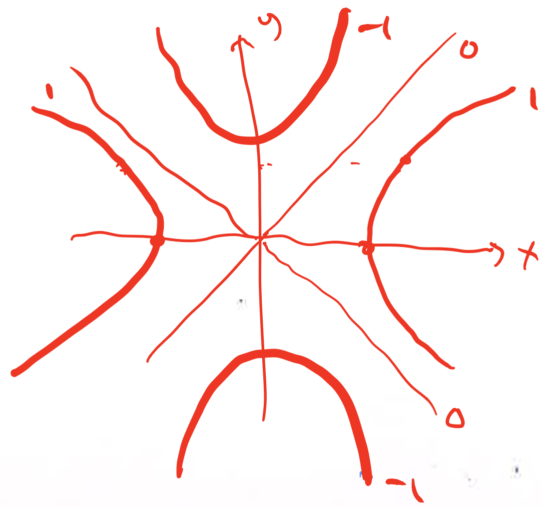
what is the graph

Level curves:

$c=0$: $x^2 - y^2 = 0$

$(x-y)(x+y) = 0$

$y = x$ or $y = -x$ straight lines.



$c=1$: $x^2 - y^2 = 1$

$(x-y)(x+y) = 1$

$x + y = \frac{1}{x-y}$

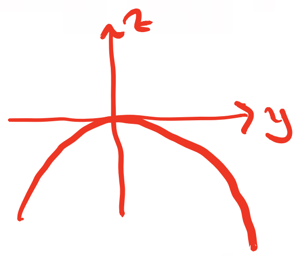
$c=-1$: $x^2 - y^2 = -1$

$y^2 - x^2 = 1$

Other cuts:

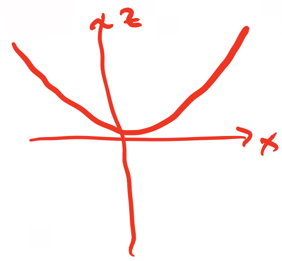
$x = 0$

$z = 0^2 - y^2 = -y^2$



$y = 0$

$z = x^2 - 0^2 = x^2$



3 Linear functions

A function is **linear** if it can be written as

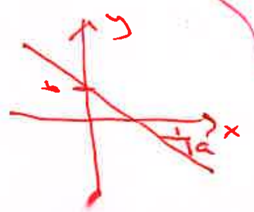
$$f(x,y) = ax + by + c$$

where a, b, c are constants.

Result: $\boxed{f \text{ is linear} \Longleftrightarrow \text{the graph of } f \text{ is a plane}}$

$f(x,y) = ax + by + c$

a plane is a surface in $\mathbb{R}^3$ (three-dim. coordinate system) which is without curvature



Compare:

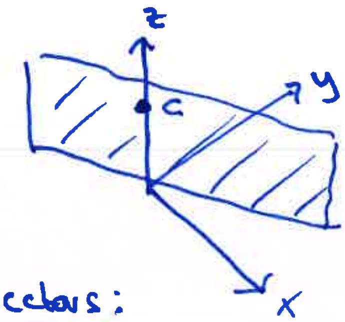
$y = f(x) = ax + b$ is linear

$\Updownarrow$

graph of f is a straight line

c = intersection with the z-axis

$f(x,y) = ax + by + c \Rightarrow f(0,0) = c$



Innerproduct / dot product of vectors:

$$\left.\begin{array}{l} \underline{u} = (u_1, u_2, u_3) \\ \underline{v} = (v_1, v_2, v_3) \end{array}\right\} \quad \underline{u} \cdot \underline{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3$$

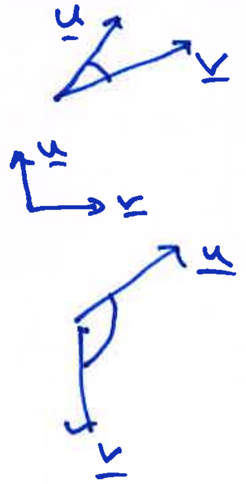
$\Updownarrow$

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Ex: $\underline{u} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$\underline{u} \cdot \underline{v} = 1 \cdot 2 + 3 \cdot (-1) + 1 \cdot 0 = -1$$

Interpretation:



$\underline{u}\cdot\underline{v} > 0$: the angle between $\underline{u}$ and $\underline{v}$ is <u>less than</u> $90°$

$\underline{u}\cdot\underline{v} = 0$: $\underline{u} \perp \underline{v}$ (the angle is $90°$)

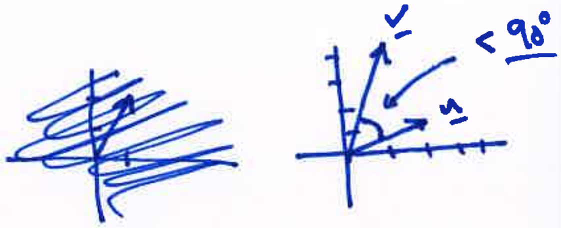
$\underline{u}\cdot\underline{v} < 0$: the angle between $\underline{u}$ and $\underline{v}$ is <u>more than</u> $90°$

Important fact: $\underline{u}\cdot\underline{v} = 0 \iff \underline{u} \perp \underline{v}$

($\underline{u}$ is normal to $\underline{v}$, the angle between $\underline{u}$ and $\underline{v}$ is $90°$)

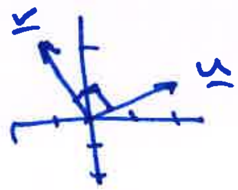
Ex: $\underline{u} = \begin{pmatrix}2\\1\end{pmatrix}$ $\underline{v} = \begin{pmatrix}1\\4\end{pmatrix}$

$\underline{u}\cdot\underline{v} = 2\cdot 1 + 1\cdot 4 = \underline{6}$



$\underline{u} = \begin{pmatrix}2\\1\end{pmatrix}$ $\underline{v} = \begin{pmatrix}-1\\2\end{pmatrix}$

$\underline{u}\cdot\underline{v} = 2\cdot(-1) + 1\cdot 2 = 0$



Let $f(x,y) = ax + by$:

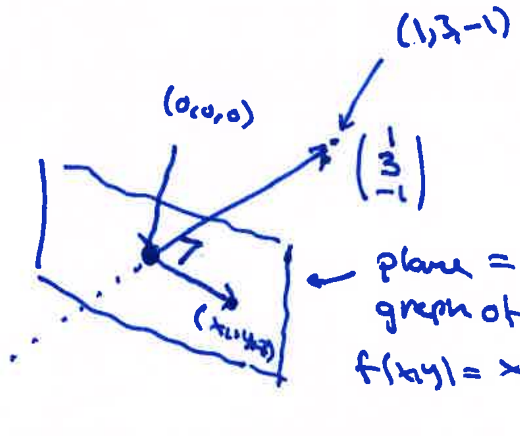
$z = ax + by \iff ax + by - z = 0$

$\begin{pmatrix}a\\b\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

normal vector of the plane

Ex: $\left.\begin{aligned} f(x,y) &= x+3y \\ z &= x+3y \\ x+3y-z &= 0 \end{aligned}\right\}$ $\begin{pmatrix}1\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

normal vector



plane = graph of $f(x,y) = x+3y$

Conclude: $f(x,y,z) = ax + by + c$ linear $\Rightarrow$ Graph of f = plane

c = z-intersection

$(a,b) \rightsquigarrow \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$ normal vector

④ Partial derivatives

Ex: $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y \cdot 1 + 0 = 3x^2 - 3y$ ← partial derivative w.r.t. x: take derivative w.r.t. x as if y is constant

$f'_y = 0 - 3x \cdot 1 + 3y^2 = -3x + 3y^2$

partial derivative w.r.t. y

x is constant

Part 2:

3. c) $f(x,y) = 4x^2 + 9y^2$, $c = -2, -1, 0, 1, 2$

General: $f(x,y) = c$

$4x^2 + 9y^2 = c$

$c > 0$: $\frac{4x^2}{c} + \frac{9y^2}{c} = 1$

$\frac{x^2}{c/4} + \frac{y^2}{c/9} = 1$

Ellipse, center $(0,0)$,

half-axis $a = \sqrt{c/4} = \sqrt{c}/2$

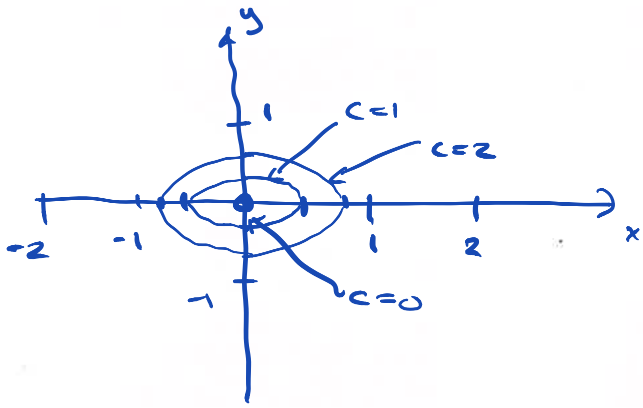
$b = \sqrt{c/9} = \sqrt{c}/3$

$c = 1$: $a = 1/2$, $b = 1/3$

$c = 2$: $a = \sqrt{2}/2$, $b = \sqrt{2}/3$

$c = -1, -2$: no points

$c = 0$: one pt. $(0,0)$



5. h) $f(x,y) = \sqrt{x^2+y^2} = \sqrt{u} = u^{1/2}$, where $u = x^2 + y^2$

$f'_x = \frac{1}{2\sqrt{u}} \cdot u'_x = \frac{1}{2\sqrt{u}} \cdot (2x + 0) = \frac{\cancel{2}x}{\cancel{2}\sqrt{u}} = \frac{x}{\sqrt{x^2+y^2}}$

$f'_y = \frac{1}{2\sqrt{u}} \cdot u'_y = \frac{1}{2\sqrt{u}} \cdot (0 + 2y) = \frac{\cancel{2}y}{\cancel{2}\sqrt{u}} = \frac{y}{\sqrt{x^2+y^2}}$

6. a) $f = \frac{1}{x+y} = \frac{1}{u} = u^{-1}$, $u = x + y$

$f'_x = -1 \cdot u^{-2} \cdot u'_x = -\frac{1}{u^2} \cdot 1 = -\frac{1}{(x+y)^2}$

$f'_y = -1 \cdot u^{-2} \cdot u'_y = -\frac{1}{u^2} \cdot 1 = -\frac{1}{(x+y)^2}$

c) $f = \frac{xy}{2x-y} = \frac{u}{v}$ $\quad u = xy \quad v = 2x-y \quad u'_x = y \quad v'_x = 2 \quad u'_y = x \quad v'_y = -1$

$$f'_x = \frac{u'_x v - u v'_x}{v^2} = \frac{y\cdot(2x-y) - xy\cdot 2}{(2x-y)^2} = \frac{\cancel{2xy} - y^2 - \cancel{2xy}}{*}$$

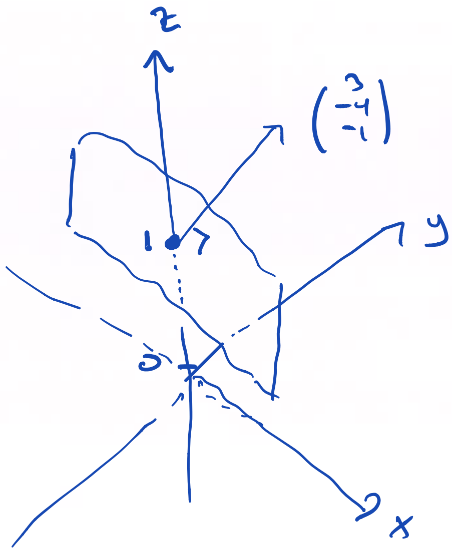
$$= \frac{-y^2}{(2x-y)^2}$$

$$f'_y = \frac{u'_y \cdot v - u \cdot v'_y}{v^2} = \frac{x(2x-y) - xy\cdot(-1)}{(2x-y)^2}$$

$$= \frac{2x^2 - \cancel{xy} + \cancel{xy}}{(2x-y)^2} = \frac{2x^2}{(2x-y)^2}$$

7.) $f(x,y) = 3x - 4y + 1$

$\boxed{\text{The graph of } f \text{ is the plane that intersects the z-axis at } z=1 \text{ and with normal vector } (3,-4,-1) \text{ or } \begin{pmatrix}3\\-4\\-1\end{pmatrix}}$



$z = 3x - 4y$

$0 = 3x - 4y - z$

$0 = \begin{pmatrix}3\\-4\\-1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix}$

2d) $\underline{v} = \begin{pmatrix}4\\7\\-3\end{pmatrix}$: $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}4\\7\\-3\end{pmatrix} = 0$

$4x + 7y - 3z = 0$

$\left(4 \;\; 7 \;\; -3 \mid 0\right)$

y, z free

$4x + 7y - 3z = 0$

$4x = -7y + 3z$

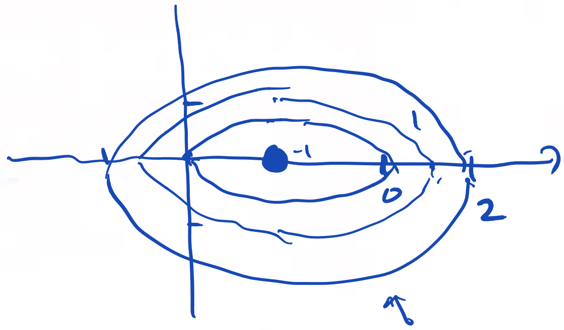
$x = -\frac{7}{4}y + \frac{3}{4}z$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-7/4\, y + 3/4\, z\\ y\\ z\end{pmatrix} = \begin{pmatrix}-7/4\, y\\ y\\ 0\end{pmatrix} + \begin{pmatrix}3/4\, z\\ 0\\ z\end{pmatrix}$$

$$= y\cdot\begin{pmatrix}-7/4\\1\\0\end{pmatrix} + z\cdot\begin{pmatrix}3/4\\0\\1\end{pmatrix}$$ with y, z free

3d) $f(x,y) = x^2 - 2x + 4y^2$

$c = -2, -1, 0, 1, 2$



$c = -2$: no pts
$c = -1$: $(1,0)$
$c = 0$: ellipse
$c = 1$: ellipse
$c = 2$: ellipse

In general:

$x^2 - 2x + 4y^2 = c \qquad | +1$

$\underbrace{x^2 - 2x + 1} + 4y^2 = c + 1$

$(x-1)^2 + 4y^2 = c + 1 \qquad | : (c+1)$

$$\frac{(x-1)^2}{c+1} + \frac{4y^2}{c+1} = 1$$

$$\frac{(x-1)^2}{c+1} + \frac{y^2}{(c+1)/4} = 1$$

$c < -1$: no pts.

$c = -1$: pt. $(1,0)$

$c > -1$:

Ellipse w/ center $(1,0)$ and half-axis

$a = \sqrt{c+1}$

$b = \sqrt{\frac{c+1}{4}} = \frac{\sqrt{c+1}}{2}$

$c + 1 > 0$ i.e. $c > -1$