Plan

1. Partial derivatives and the Hessian
2. Tangents of level curves
3. The gradient and directional derivatives

1 Partial derivatives

$f(x,y)$: function in two variables, $z = f(x,y)$

<u>Defn:</u>

$$f'_x = f'_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{dz}{dx}$$ x var. y const.

$$f'_y = f'_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{dz}{dy}$$ x const. y var.

<u>Examples:</u>

i) $f(x,y) = \underline{x^2-4x} \; \underline{+y^2+2y}$

$f'_x = 2x - 4 + 0 = \underline{2x-4}$

$f'_y = 0 + 2y + 2 = \underline{2y+2}$

ii) $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y\cdot 1 + 0 = \underline{3x^2-3y}$

$f'_y = 0 - 3x\cdot 1 + 3y^2 = \underline{-3x+3y^2}$

iii) $f(x,y) = \sqrt{x^2+y^2}$

$= \sqrt{u}, \quad u = \underline{x^2+y^2}$

$= u^{1/2}$

$(u^{1/2})' = \frac{1}{2}u^{-1/2} = \frac{1}{2}\cdot\frac{1}{\sqrt{u}}$

$f'_x = \frac{1}{2\sqrt{u}} \cdot u'_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{u}} = \underline{\frac{x}{\sqrt{x^2+y^2}}}$

$f'_y = \frac{1}{2\sqrt{u}} \cdot u'_y = \frac{\cancel{2}y}{\cancel{2}\sqrt{u}} = \underline{\frac{y}{\sqrt{x^2+y^2}}}$

Interpretation of partial derivatives

What does $f'_x(a,b)$ and $f'_y(a,b)$ mean?

Ex: $f(x,y) = x^3 - 3xy + y^3$ $\quad f'_x = 3x^2 - 3y$

$f'_y = -3x + 3y^2$

$(a,b) = (2,1)$:

$f'_x(2,1) = \underline{9} \quad f'_y(2,1) = \underline{-3}$

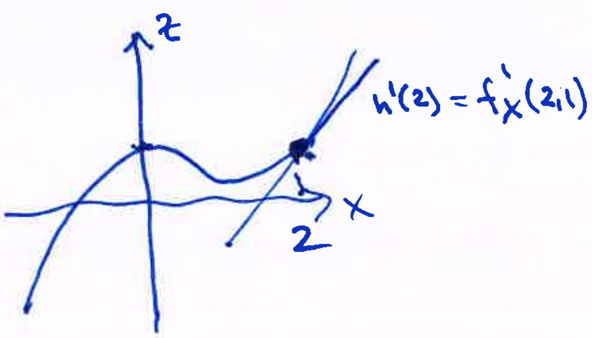
i) $\underline{y=1}$:

$h(x) = f(x,1)$

$= x^3 - 3x + 1$

$h'(x) = 3x^2 - 3$

$h'(2) = 3\cdot 2^2 - 3 = 9 = f'_x(2,1)$

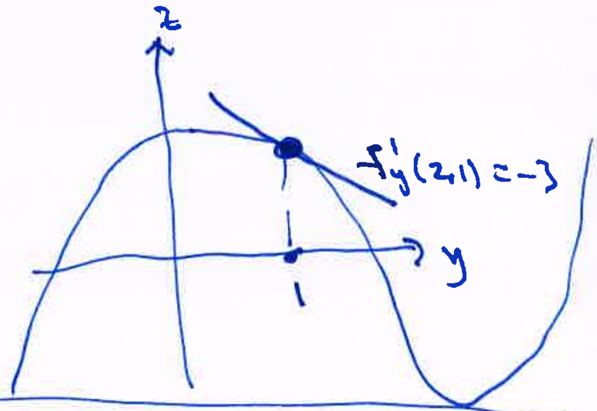


ii) $\underline{x=2}$:

$h(y) = f(2,y) = \underline{8 - 6y + y^3}$

$h'(y) = -6 + 3y^2$

$h'(1) = -3 = f'_y(2,1)$



$\boxed{\text{\underline{Defn}: A \underline{stationary pt} for } f \text{ is a point } (x,y) \text{ such that } f'_x(x,y) = f'_y(x,y) = 0.}$

The Hessian of f:

Defn: $H(f)(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix}$ is the Hessian matrix of f.

Ex: $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y$ $\quad f''_{xx} = (3x^2-3y)'_x = 6x \quad f''_{xy} = (3x^2-3y)'_y = -3$

$f'_y = -3x + 3y^2$ $\quad f''_{yx} = (-3x+3y^2)'_x = -3 \quad f''_{yy} = (-3x+3y^2)'_y = 6y$

$H(f)(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ $\quad H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

the Hessian matrix of f at $(x,y)=(1,1)$

Results:

① If (x,y) is a max/min of f, then it is a stationary pt.

② For each stationary pt. (x^*,y^*) of f, we can use $H(f)(x^*,y^*)$ to classify it as local max, local min or saddle pt.

2) Tangents of level curves

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Level curves: $f(x,y) = c$ ← horizontal cut $z = c$ through the graph of f

→ all pts on the graph of f in height c

$x^2 - 2x + y^2 + 4y = c \quad | +1 +4$

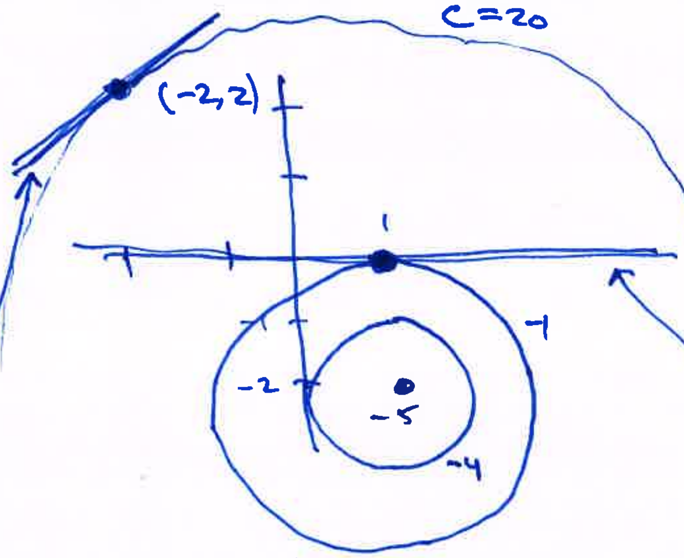
$x^2 - 2x + 1 + y^2 + 4y + 4 = c + 5$

$$\boxed{(x-1)^2 + (y+2)^2 = c+5}$$

$c > -5$: circle with center $(1,-2)$ and radius $r = \sqrt{c+5}$

$c = -5$: pt. $(1,-2)$

$c < -5$: no pts



Ex: $(x,y) = (-2,2)$ $\quad f(-2,2) = 20$

$y - 2 = k \cdot (x+2)$

$y - 2 = 3/4\,(x+2)$

$y = \frac{3}{4}x + \frac{3}{2} + 2$

$y = \frac{3}{4}x + \frac{7}{2}$

Ex: $(x,y) = (1,0)$ at $f(x,y) = -1$:

Find the tangent line of $f(x,y) = -1$ at $(1,0)$:

$y - y_0 = k \cdot (x - x_0)$

$y - 0 = k \cdot (x - 1)$

$k = y'(x_0,y_0) = y'(1,0)$

$y - 0 = 0 \cdot (x-1)$

$y = 0$

In the examples:

$x^2 - 2x + y^2 + 4y = -1$

Implicit derivation:

$2x - 2 + 2y \cdot y' + 4 \cdot y' = 0$

$(2x-2) + (2y+4)\, y' = 0$

$$\frac{(2y+4)\, y'}{2y+4} = \frac{-(2x-2)}{2y+4}$$

$$y' = -\frac{2x-2}{2y+4}$$

$$y'(1,0) = -\frac{2\cdot 1 - 2}{2\cdot 0 + 4} = -\frac{0}{4} = \underline{0}$$

$$y'(-2,2) = -\frac{2\cdot(-2)-2}{2\cdot 2+4} = \frac{6}{8} = \underline{3/4}$$

In general:

$f(x,y) = c$

Implicit derivation

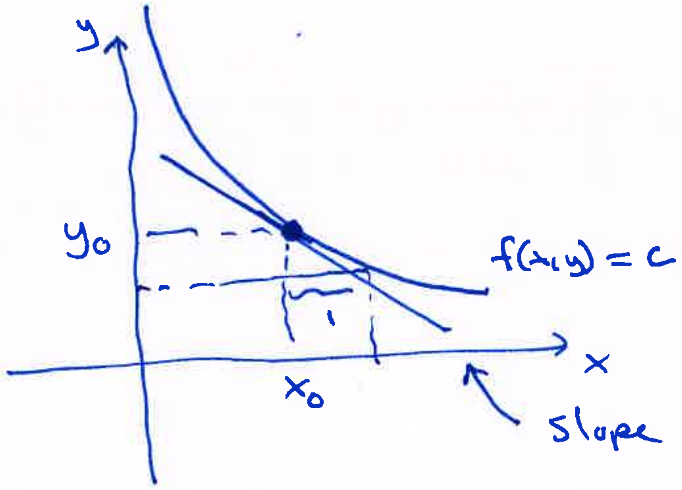
$f'_x + f'_y \cdot y' = 0$

$$\frac{f'_y\, y'}{f'_y} = \frac{-f'_x}{f'_y}$$

$$\boxed{y' = -\frac{f'_x}{f'_y}}$$

Formula for the slope of a tangent to the level curve $f(x,y) = c$.

Application



$$\text{slope} = -\frac{f'_x}{f'_y}$$

how much less would you use of y if you use one unit more of x

③ Gradients:

Defn: The <u>gradient</u> of $f(x,y)$ is the vector

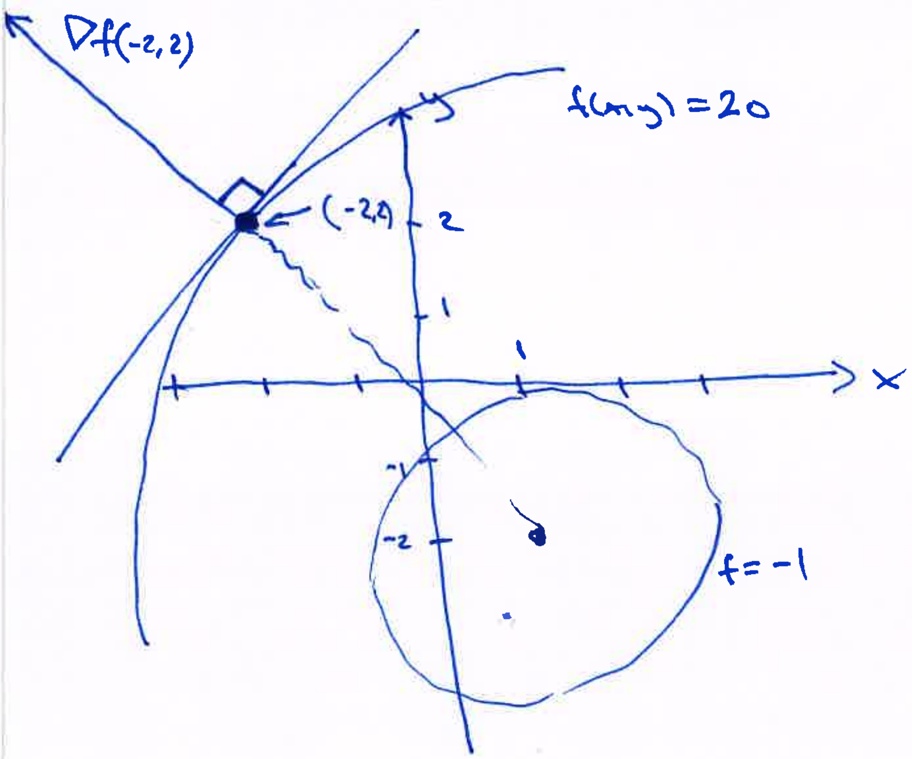
(∇ = nabla)

$$\nabla f = \begin{pmatrix} f'_x(x,y) \\ f'_y(x,y) \end{pmatrix} = \left(f'_x(x,y),\ f'_y(x,y) \right)$$

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

$$\left.\begin{array}{l} f'_x = 2x - 2 = 2x-2 \\ f'_y = 2y + 4 = 2y+4 \end{array}\right\} \quad \nabla f = \begin{pmatrix} 2x-2 \\ 2y+4 \end{pmatrix}$$

$$\nabla f(-2,2) = \underline{\begin{pmatrix} -6 \\ 8 \end{pmatrix}}$$



<u>Tangent line:</u>

$$y = \frac{3}{4}x + \frac{7}{2}$$

$f(x,y) = 20$

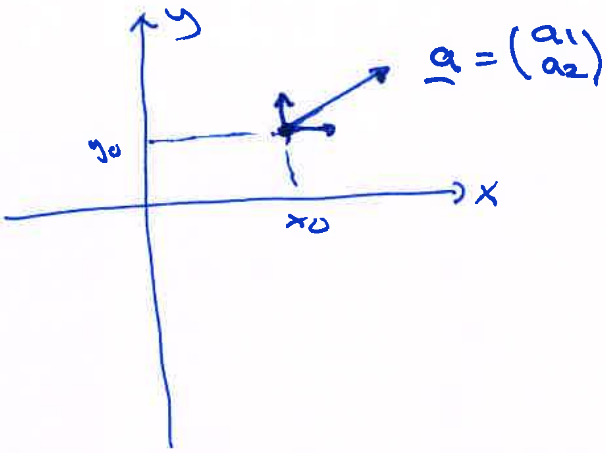
$(x-1)^2 + (y+2)^2 = 5^2$

<u>Properties of the gradient:</u>

① The gradient is normal (90°) to the tangent of the level curve

② The gradient points in the direction where f grows fastest.

Directional derivatives



Defn: Directional derivative of f

$$f'_{\underline{a}} = a_1 \cdot f'_x + a_2 \cdot f'_y$$
$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \underline{a} \cdot \nabla f$$

$$\boxed{f'_{\underline{a}} = \underline{a} \cdot \nabla f}$$

Ex: $\underline{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$f(x,y) = x^3 + 3xy + y^3$$

$$\left.\begin{aligned} f'_x &= 3x^2 + 3y \\ f'_y &= 3x + 3y^2 \end{aligned}\right\} \quad \nabla f = \begin{pmatrix} 3x^2 + 3y \\ 3x + 3y^2 \end{pmatrix}$$

$$f'_{\underline{a}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3x^2 + 3y \\ 3x + 3y^2 \end{pmatrix} = 2(3x^2 + 3y) + 1 \cdot (3x + 3y^2)$$
$$= \underline{6x^2 + 6y + 3x + 3y^2}$$

Part 2:

1. $f(x,y) = x^2 - 2x + 4y^2$

a) $f(x,y) = c$

$x^2 - 2x + 4y^2 = c \quad | +1$

$x^2 - 2x + 1 + 4y^2 = c + 1$

$$\boxed{(x-1)^2 + 4y^2 = c+1}$$

$$\frac{(x-1)^2}{c+1} + \frac{4y^2}{c+1} = \frac{c+1}{c+1}$$

$$\frac{(x-1)^2}{c+1} + \frac{y^2}{(c+1)/4} = 1$$

$c > -1$: Ellipse, center $(1,0)$, half-axis $a = \sqrt{c+1}$ ✓, $b = \sqrt{c+1}/2$ ✓

$$\boxed{y' = -\frac{f'_x}{f'_y} = -\frac{2x-2}{8y}}$$

b) $(x,y) = (1,1)$: $f(1,1) = 3$

$y - 1 = k \cdot (x-1)$

$y - 1 = -\frac{2-2}{8 \cdot 1}(x-1)$ $\qquad \underline{\underline{y = 1}}$

$(x,y) = (2, \frac{1}{2})$: $f(2, \frac{1}{2}) = 1$

$y - \frac{1}{2} = k \cdot (x-2)$

$y - \frac{1}{2} = -\frac{2 \cdot 2 - 2}{8 \cdot \frac{1}{2}}(x-2)$

$y - \frac{1}{2} = -\frac{1}{2}x + 1$

$$\underline{\underline{y = -\frac{1}{2}x + \frac{3}{2}}}$$

c) $\nabla f(1,1) = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

$\nabla f(2, \frac{1}{2}) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

f will increase along the gradients

d) Yes, a minimum at $(1,0)$: $f(1,0) = \underline{-1}$

$c > -1$: $f(x,y) = c$ ellipse

$c = -1$: $f(x,y) = -1$ pt $(1,0)$

$c < -1$: $f(x,y) = c$ no pts.

No maximum

3.c) $f(x,y) = 4x^2 - 6xy + 9y^2$

$f'_x = \underline{8x - 6y}$ $\quad f''_{xx} = 8 \quad f''_{xy} = -6$

$f'_y = \underline{-6x + 18y}$ $\quad f''_{yx} = -6 \quad f''_{yy} = 18$

$$H(f) = \underline{\underline{\begin{pmatrix} 8 & -6 \\ -6 & 18 \end{pmatrix}}} \qquad H(f)(1,1) = \underline{\underline{\begin{pmatrix} 8 & -6 \\ -6 & 18 \end{pmatrix}}}$$

Notice: We always have $\boxed{f''_{xy} = f''_{yx} \implies H(f) \text{ is symmetric}}$

4.b) $f(x,y) = x^2 + y^2$

$$\left.\begin{aligned} f'_x &= 2x \\ f'_y &= 2y \end{aligned}\right\} \quad \nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \implies \nabla(f)(1,1) = \underline{\underline{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}}$$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : \quad f'_{\underline{a}} = \underline{a} \cdot \nabla f = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$= a_1 \cdot 2x + a_2 \cdot 2y$$

$$f'_{\underline{a}}(1,1) = \underline{\underline{2a_1 + 2a_2}}$$

5. a) $f(x,y) = 2x + 3y$

$\underline{(x,y) = (1,1)}$: $f(1,1) = 5$

$$\left.\begin{aligned} f'_x &= 2 \\ f'_y &= 3 \end{aligned}\right\} \quad y' = -\frac{2}{3}$$

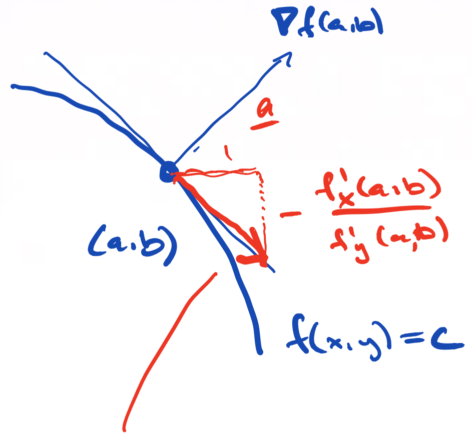
$$y - 1 = k \cdot (x-1)$$

$$y - 1 = -\frac{2}{3}(x-1)$$

$$y = -\frac{2}{3}x + \frac{2}{3} + 1$$

$$\underline{\underline{y = -\frac{2}{3}x + \frac{5}{3}}}$$

6.



Tangent line:

$$y-b=-\frac{f'_x(a,b)}{f'_y(a,b)}\cdot(x-a)$$

Gradient:

$$\nabla f(a,b)=\begin{pmatrix} f'_x(a,b) \\ f'_y(a,b) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -\frac{f'_x(a,b)}{f'_y(a,b)} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} f'_x(a,b) \\ f'_y(a,b) \end{pmatrix}}_{\nabla f(a,b)}\cdot\begin{pmatrix} 1 \\ -\frac{f'_x(a,b)}{f'_y(a,b)} \end{pmatrix}$$

This means that ∇f is normal to the tangent line. $\longrightarrow$

$$=f'_x(a,b)\cdot 1+\cancel{f'_y(a,b)}\cdot\left(-\frac{f'_x(a,b)}{\cancel{f'_y(a,b)}}\right)$$

$$=f'_x(a,b)-f'_x(a,b)=\underline{0}$$

Directional derivative of f in the direction $\underline{a}=\nabla f(a,b)$:

$$f'_{\underline{a}}(a,b)=\underline{a}\cdot\nabla f(a,b)$$

$$=\nabla f(a,b)\cdot\nabla f(a,b)$$

$$=\begin{pmatrix} f'_x(a,b) \\ f'_y(a,b) \end{pmatrix}\cdot\begin{pmatrix} f'_x(a,b) \\ f'_y(a,b) \end{pmatrix}$$

$$=f'_x(a,b)^2+f'_y(a,b)^2\geq 0$$

f will increase in the direction of the gradient