Plan

1. Stationary points
2. Second derivative test
3. Global maxima and minima

Extra lecture Monday

Problems – tell me if there are specific things you want me to go through.

Problem: max/min $f(x,y)$

① Stationary points

Defn: A stationary pt for f is a point (x^*,y^*) such that $\left\{\begin{array}{l} f'_x(x^*,y^*)=0 \\ f'_y(x^*,y^*)=0 \end{array}\right\}$ ← first order conditions (FOC)

Ex: $f(x,y) = x^2-2x+y^2+4y$

$f'_x = 2x-2 = 0 \qquad x=1$

$f'_y = 2y+4 = 0 \qquad y=-2$

Stat pts: $(x,y) = (1,-2)$

Ex: $f(x,y) = x^3+3xy+y^3$

$f'_x = 3x^2+3y = 0 \Rightarrow x^2+y=0 \Rightarrow y=-x^2$

$f'_y = 3x+3y^2 = 0 \qquad x+y^2=0$

$x+(-x^2)^2=0$

$x+x^4=0$

$x(1+x^3)=0$

$x=0,\ y=0$ or $x^3=-1$

$x=\sqrt[3]{-1}=-1 \quad y=-1$

Stationary pts: $(x,y) = (0,0),\ (-1,-1)$

Result: If (x^*, y^*) is a maximum or minimum of f, then we have either

critical points $\begin{cases} \text{i)} & (x^*, y^*) \text{ is a stationary point} \\ \text{ii)} & (x^*, y^*) \text{ is a point where } f'_x \text{ or } f'_y \text{ does not exist} \end{cases}$

iii) (x^*, y^*) is a boundary point of D_f

Candidates for max/min = stationary pts + exceptions (ii or iii)

Ex: $f(x,y) = x^3 + 3xy + y^3$, $D_f = \mathbb{R}^2$

$f'_x = 3x^2 + 3y$

$f'_y = 3x + 3y^2$

i) Stationary pts: $(0,0)$, $(-1,-1)$

ii) Other critical pts: none

iii) Boundary pts: none

Ex: $f(x,y) = \sqrt{x+y}$, $D_f = \{(x,y) : x+y \geq 0\}$

$= \sqrt{u}$, $u = x+y$

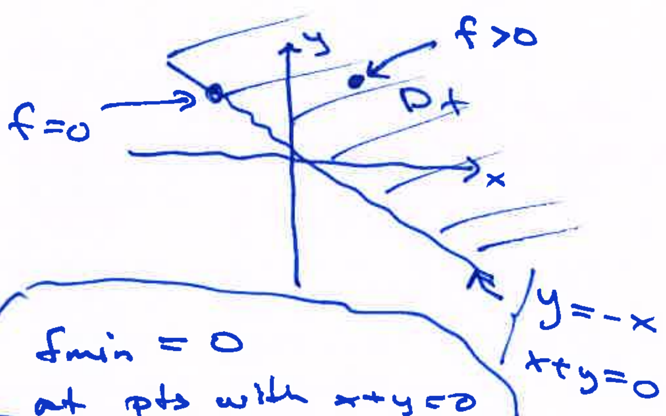
$f'_x = \frac{1}{2\sqrt{u}} \cdot u'_x = \frac{1}{2\sqrt{x+y}} = 0$

$f'_y = \frac{1}{2\sqrt{u}} u'_y = \frac{1}{2\sqrt{x+y}} = 0$

i) Stationary pts: none

ii) Other critical pts: pts where $x+y=0$ = line

$x+y \geq 0$: $y \geq -x$



$f_{min} = 0$ at pts with $x+y=0$

iii) Boundary pts of D_f: pts with $x+y=0$ = line

② Second derivative test

Ex: $f(x,y) = x^3 + 3xy + y^3$

$f'_x = 3x^2 + 3y$ Crit pts: $(0,0)$, $(-1,-1)$

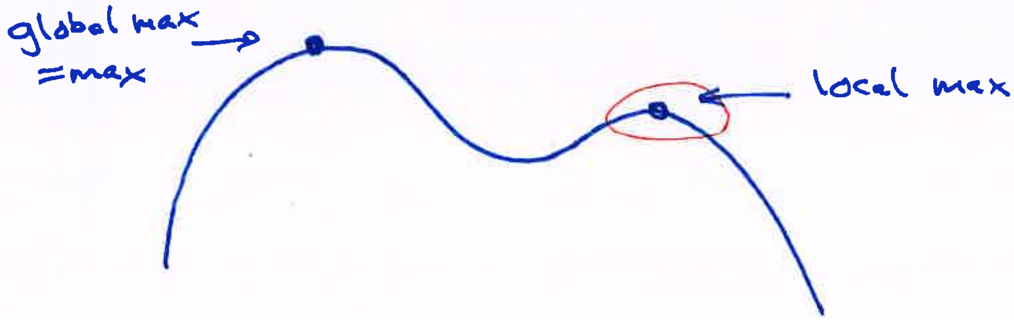
$f'_y = 3x + 3y^2$ $f(0,0) = 0$ $f(-1,-1) = 1$

Defn: (x^*,y^*) is a max (global max) for f if

$f(x^*,y^*) \geq f(x,y)$ for all (x,y) in D_f.

(x^*,y^*) is a local max for f if

$f(x^*,y^*) \geq f(x,y)$ for all (x,y) close to (x^*,y^*)



Similar for min / local min.

Defn: A saddle pt for f is a stationary pt. that is neither local max nor local min.

Second derivative test:

Let (x^*,y^*) be a stationary pt of f. We compute

$$H(f)(x^*,y^*) = \begin{pmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{xy}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

and look at $\det H(f)(x^*,y^*) = AC - B^2$

$\operatorname{tr} H(f)(x^*,y^*) = A + C$

i) If $AC - B^2 > 0$, $A + C > 0 \Rightarrow (x^*,y^*)$ is local min

ii) If $AC - B^2 > 0$, $A + C < 0$ (x^*,y^*) is local max

iii) If $AC - B^2 < 0$ (x^*,y^*) is saddle point

Ex: $f(x,y) = x^3 + 3xy + y^3$

$f'_x = 3x^2 + 3y = 0$

$f'_y = 3x + 3y^2 = 0$ $\Rightarrow$ Stat. pts = Cand. pts:

$(x,y) = (0,0),\ (-1,-1)$

$f=0 \qquad f=1$

Classify stationary pts:

$H(f) = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$

$f''_{xx} = 6x \qquad f''_{xy} = 3$

$f''_{yx} = 3 \qquad f''_{yy} = 6y$

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} \overset{A}{\overset{\shortparallel}{0}} & \overset{B}{\overset{\shortparallel}{3}} \\ 3 & \underset{C}{\underset{\shortparallel}{0}} \end{pmatrix}$

det $= 0 - 9 = -9 = AC - B^2$

tr $= 0 + 0 = 0$

det < 0 : $(0,0)$ saddle pt

$(-1,-1)$: $H(f)(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$

det $= AC - B^2 = 36 - 9 = 27 > 0$

tr $= -6 - 6 = -12 < 0$

$\uparrow$

$A + C$

$\Downarrow$

$(-1,-1)$ local max

What about max/min $f(x,y) = x^3 + 3xy + y^3$:

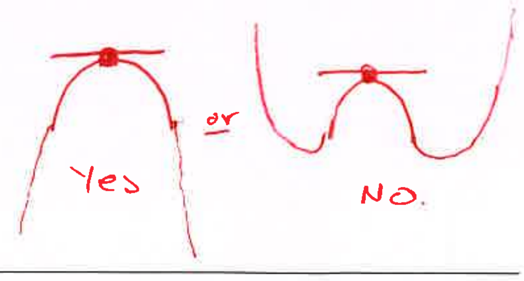
Candidate pts: $(0,0)$, $(-1,-1)$

$f(0,0) = 0 \qquad f(-1,-1) = 1$

saddle pt $\qquad$ local max

Global min: No.

Global max: Maybe $(-1,-1)$



Second derivative test:

$$H(f)(x^*,y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad \begin{aligned} \det &= AC-B^2 \\ tr &= A+C \end{aligned}$$

i) $AC-B^2>0$, $A+C>0 \Rightarrow$ local min
ii) $AC-B^2>0$, $A+C<0 \Rightarrow$ local max
iii) $AC-B^2<0$ $\Rightarrow$ saddle pt

a) What is missing from this list:

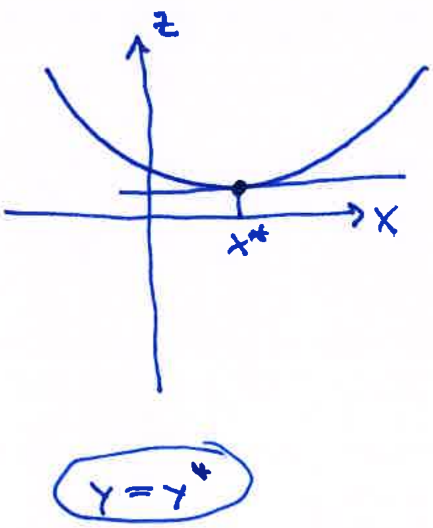
$AC-B^2=0$: The test is inconclusive. We must use the definition.

$AC-B^2>0$: $AC>B^2\geq 0 \Rightarrow AC>0$

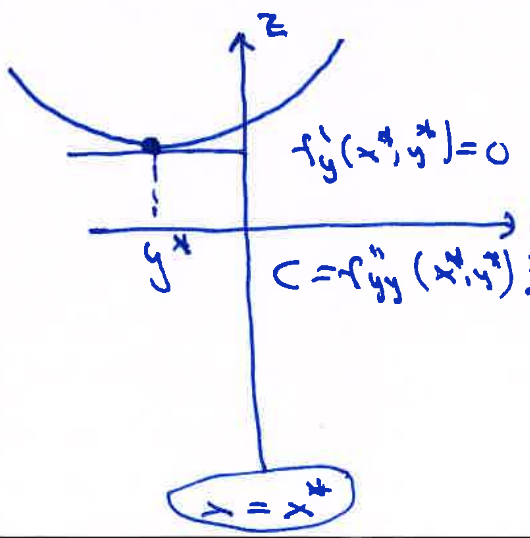
$A+C>0 \longleftrightarrow$ i) $A>0, C>0$
or
$A+C<0 \longleftrightarrow$ ii) $A<0, C<0$

b) Explanation:

Consider the case $A+C>0$ and $AC-B^2>0$:
$(A>0, C>0)$



$f'_x(x^*,y^*)=0$
$A=f''_{xx}(x^*,y^*)>0$



$f'_y(x^*,y^*)=0$
$C=f''_{yy}(x^*,y^*)>0$

(3) Global max / min.

(x^*, y^*) global max (min) for f $\Rightarrow$ (x^*, y^*) local max (min) for f in the list of candidate pts.

Ex: $f = x^3 + 3xy + y^3$

Cand. pts: $(0,0)$, $f=0$, saddle pt, ; $(-1,-1)$, $f=1$, local max

maybe global max

Question: Can $f(x,y) > 1$?

$f(x,y) = x^3 + 3xy + y^3$

$y=0$: $f(x,0) = x^3$

$f(2,0) = 2^3 = 8 > 1$

No, $(-1,-1)$ is not global max

$\Downarrow$

no global max.

Part 2: Extra lecture on Monday
Will cover Linear Approximation

Problem 4. Find global max/min.

a) $f(x,y) = 2x + 3y$

$f'_x = 2 = 0$, $f'_y = 3 = 0$ } no stat. pts.

No cand. pts.

$\Downarrow$

no max/min

b) $f = x^2 + y^2$

$f'_x = 2x = 0$, $f'_y = 2y = 0$ } $(x,y) = (0,0)$

stat. pts = cand. pts.

$H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ det $= 4 > 0$, tr $= 4 > 0$ } $(0,0)$ local min

No max.

Min: $f(0,0) = 0$ local min

$f(x,y) = x^2 + y^2 \geq 0$ for all (x,y)

$\Downarrow$

$f(0,0) = 0$ global min

f) $f(x,y) = y^2 - x^3 + 3x$

$f'_x = -3x^2 + 3 = 0$ $x^2 = 1$ $x = \pm 1$
$f'_y = 2y = 0$ $y = 0$

Cand. pts: $(\pm 1, 0)$

$$H(f) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix}$$

$$H(f)(1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \quad \det = -12 < 0 \quad \text{saddle pt}$$

$$H(f)(-1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \quad \det = 12 > 0, \ \operatorname{tr} = 8 > 0 \quad \text{local min}$$

$$f(-1,0) = -2$$

Global max: No

Global min: $f(x,y) = y^2 - x^3 + 3x$

$y=0$: $f(x,0) = -x^3 + 3x$

$f(3,0) = -27 + 9 = -18 < -2$

No global min.

h) $f(x,y) = \ln(u)$, $u = x^2y^2 - x^2 - y^2 + 3$

$$f'_x = \frac{1}{u} \cdot u'_x = \frac{2xy^2 - 2x}{u} = 0 \quad \Rightarrow \quad 2x(y^2-1) = 0$$

$$f'_y = \frac{1}{u} \cdot u'_y = \frac{2x^2y - 2y}{u} = 0 \qquad 2y(x^2-1) = 0$$

$x=0$, $y=0$; $y^2=1$, $y=\pm 1$, $x^2=1$, $x=\pm 1$

Cand. pts: $(x,y) = \underset{u=3}{(0,0)}, \ \underset{u=2}{(\pm 1, \pm 1)}$

(h) Alt I: Second derivative test

$$f''_{xx} = \left[\frac{2x(y^2-1)}{u}\right]'_x = \frac{2(y^2-1)\cdot u - 2x(y^2-1)\cdot 2x(y^2-1)}{u^2} \quad \checkmark$$

$$f''_{xy} = \left[\frac{2x(y^2-1)}{u}\right]'_y = \frac{2x \cdot 2y \cdot u - 2x(y^2-1)\cdot 2y(x^2-1)}{u^2}$$

$$f''_{yy} = \left[\frac{2y(x^2-1)}{u}\right]'_y = \frac{2(x^2-1)\cdot u - 2y(x^2-1)\cdot 2y\cdot(x^2-1)}{u^2}$$

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} 2\cdot(-1)\cdot 3/3^2 & 0 \\ 0 & 2(-1)\cdot 3/3^2 \end{pmatrix} = \begin{pmatrix} -2/3 & 0 \\ 0 & -2/3 \end{pmatrix}$

$\det = 4/9 > 0$

$\operatorname{tr} = -4/3 < 0$

local max

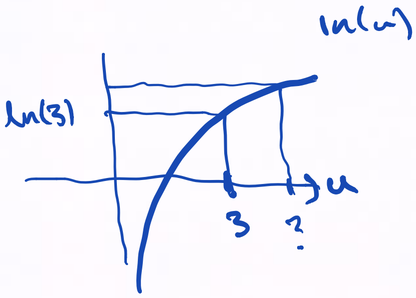
$(\pm 1, \pm 1)$: $H(f)(\pm 1, \pm 1) = \begin{pmatrix} 0 & 4(\pm 1)\cdot 2/2^2 \\ \frac{4(\pm 1)\cdot 2}{2^2} & 0 \end{pmatrix}$

$\det = 0^2 - (\pm 2)^2$
$= -4 < 0$

saddle pts

Global min: No.

Global max: $f(0,0) = \ln(3)$
local max



$f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

Can $x^2y^2 - x^2 - y^2 + 3 > 3$?

$x=y$: $x^4 - 2x^2 + 3$

$x=2$: $2^4 - 2\cdot 2^2 + 3 = 11 > 3$

$f(2,2) = \ln(11) > \ln(3)$

No max.