Plan

1 Optimization problems without constraints
2 Optimization problems with constraints
3 Extreme value theorem

Extra lectures/problem sessions:

Lecture 32:
Wed May 26

Problem session
Thu May 27

+ one lecture more

① Optimization without constraints

max/min $f(x,y)$

Method: ① Find candidate points ← all points that could possibly be max/min for f

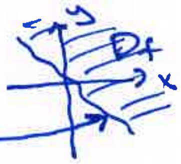
a) Stationary points:

$f'_x = 0,\ f'_y = 0$ ← FOC

b) Other critical points:

f'_x or f'_y does not exist

c) Boundary pts for D_f



⇓

List of candidate pts. Compute f for each candidate pt.

② Classify the candidate points as local max, local min or saddle point

a) Second derivative test: $H(f)(x^*,y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ → det $= AC - B^2$ → tr $= A + C$

det > 0, tr > 0 ⇒ (x^*,y^*) local min

det > 0, tr < 0 ⇒ (x^*,y^*) local max

det < 0 ⇒ saddle pt

b) Exceptions:

* stationary pt with $AC - B^2 = 0$
* other critical pts or boundary pts of D_f

⇒ Use the defn. of local max/local min.

③ Determine if the best candidate for max/min is global max/min.

- Use defn. of max/min.
- Trick: Look at cuts, curve, and see what happens to f there.

Useful tool: Wolfram Alpha
plot x^2 + xy + y^2 = 2

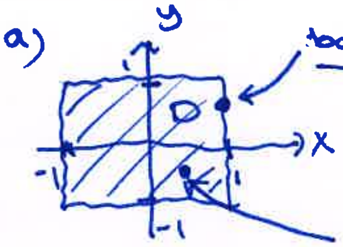
② Optimalization with constraints

Ex: a) max $\underbrace{f(x,y) = x^2+y^2}_{\text{objective fn.}}$ when $\underbrace{-1 \le x \le 1, -1 \le y \le 1}_{\text{constraints}}$

b) min $f(x,y) = xy$ when $\underbrace{2x+3y=6}_{\text{constraint}}$

c) max $f(x,y) = x+3y$ when $\underbrace{x^2+y^2=10}_{\text{constraint}}$

a)



boundary pt of D

interior pt of D

D = set of admissible points = all points satisfying all constraints

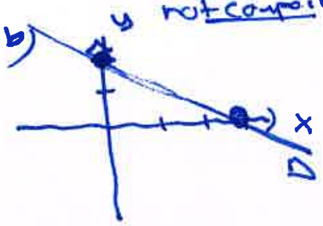
$D = \{(x,y): -1 \le x,y \le 1\}$

$D: -1 \le x,y \le 1$

closed and bounded = compact

b)

not bounded
⇒ not compact



$D: 2x+3y=6$
a line

$x=0: y=2$

$y=0: x=3$

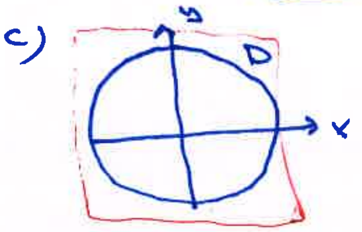
$3y = 6-2x$

$y = 2 - \frac{2}{3}x$

$\boxed{\text{All pts are boundary pts}}$

c)

bounded ⇒ compact



$D: x^2+y^2=10$
circle
with $r=\sqrt{10}$

Only boundary points

③ Extreme value theorem

Compact sets: D is a subset of $\mathbb{R}^2$

Defn: D is called compact if it is

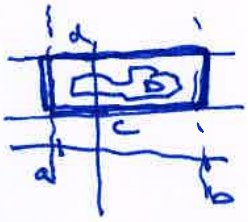
i) closed: all boundary pts of D are included in D.

$= \ \leq \ \geq$: closed $<\ >$: not closed

ii) bounded: there are number a, b, c, d such that

$$\boxed{\begin{array}{l} a \leq x \leq b \\ c \leq y \leq d \end{array}}$$

for all pts (x,y) in D.

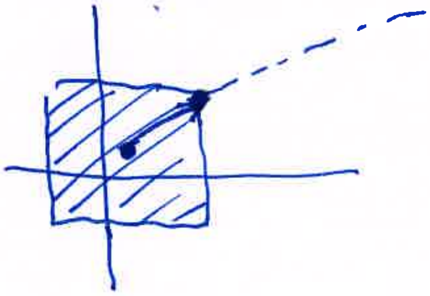


Extreme Value theorem

If f is a cont. function on a compact set D, the f attains a max and a min on D.

max/min $f(x,y)$ under some constraints $\Rightarrow$ there is max/min

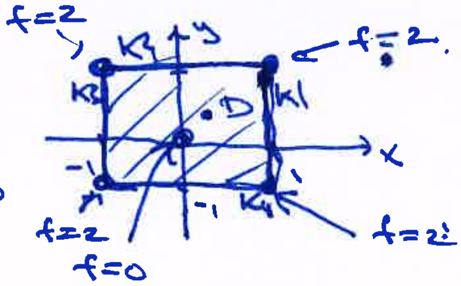
f cont D compact



D compact

Ex: max $f(x,y) = x^2 + y^2$ when $-1 \le x, y \le 1$

D = all adm. pts $= \{(x,y) : -1 \le x, y \le 1\}$



Method:

① Check if EVT applies (extreme value theorem)

② Candidate pts:

a) Interior pts that are stationary

b) Interior pts that are other critical pts

c) Boundary pts of D

When D is compact, then the max is the candidate pt with the highest f-value.

Similar Computations for K3, K4:

max $f(x,y) = \underline{\underline{2}}$ on D

Ex:

① f is cont.
D compact? Yes.
$\Rightarrow$ there is a max

② Candidate pts

a) $f'_x = 2x = 0$, $f'_y = 2y = 0$ $\Big\}$ $x = 0$, $y = 0$

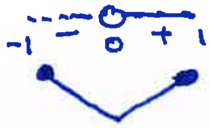
$(x,y) = (0,0)$ $f = 0$
interior pt

b) no such pts

c) Look at K1, K2, K3, K4:

K1: $x = 1$, $-1 \le y \le 1$ $f = f(1,y) = 1^2 + y^2 = y^2 + 1$

$f'_y = 2y$



$f(1,-1) = 2$
$f(1,1) = 2$

Candidates on K1: $(1, \pm 1)$ $f = 2$

K2: $y = 1$, $-1 \le x \le 1$ $f = f(x,1) = x^2 + 1$
$f'_x = 2x$

Candidates on K2: $(\pm 1, 1)$ $f = 2$

K3: $x = -1$

$-1 \leq y \leq 1$

$f = f(-1,y) = (-1)^2 + y^2 = y^2+1$

$f'_y = 2y$

Candidates on K3: $(-1,\pm 1)$ $f = 2$

K4: $y = -1$

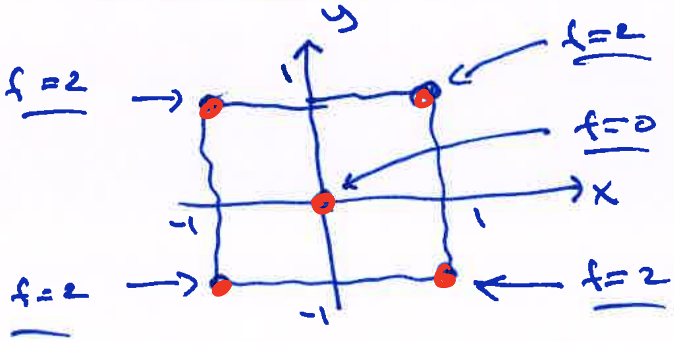
$-1 \leq x \leq 1$

$f = f(x,-1) = x^2 + (-1)^2 = x^2+1$

$f'_x = 2x$

Candidates on K4: $(\pm 1,-1)$ $f = 2$

All candidate pts:

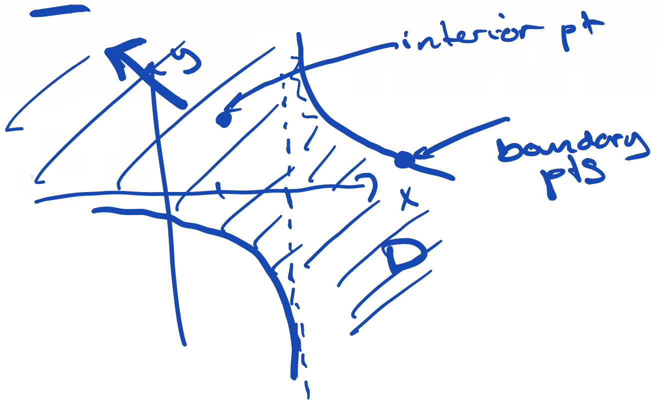


$f_{max} = 2$

at $(\pm 1, \pm 1)$

Part 2.

1. D: $y(x-2) \boxed{\le} 3$



$y(x-2) = 3$: $\quad y = \frac{3}{x-2}$

$y(x-2) < 3$:

$x > 2$: $\quad y < \frac{3}{x-2}$

$x < 2$: $\quad y > \frac{3}{x-2}$

$x = 2$: all y

closed, not bounded

$\Rightarrow$ not compact.

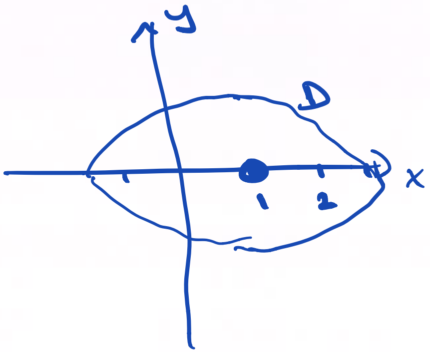
2. g) D: $x^2 - 2x + 4y^2 \boxed{=} 4 \quad |+1$

$$(x-1)^2 + 4y^2 = 5$$

$$\frac{(x-1)^2}{5} + \frac{y^2}{5/4} = 1$$

ellipse, center $(1,0)$

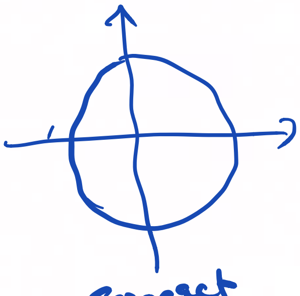
halfaxes $a = \sqrt{5}$, $b = \sqrt{5}/2$



compact: Yes

m) D: $\sqrt{x^2+y^2} = 3$

$x^2+y^2 = 9$ circle, $r = 3$



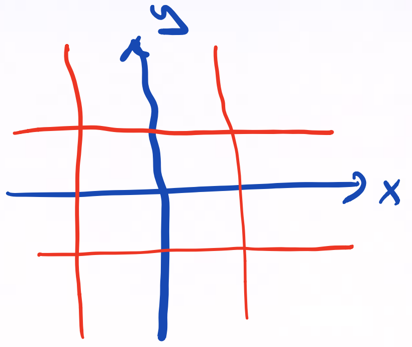
compact

o) D: $x^2y^2 - x^2 - y^2 + 1 = 0$

$(x^2 - 1)(y^2 - 1) = 0$

$x^2 = 1$ or $y^2 = 1$

$x = \pm 1$ or $y = \pm 1$ $\Big\}$ union of four lines



closed, not bounded
not compact.

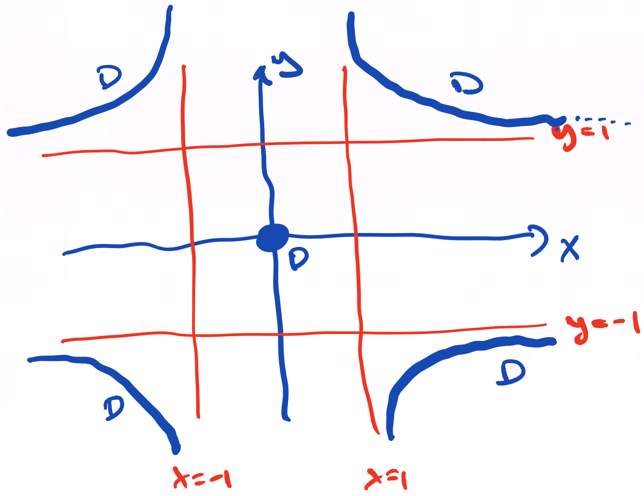
p) D: $x^2y^2 - x^2 - y^2 + 1 \;\textcircled{=}\; 1$

$(x^2-1)(y^2-1) = 1$

$y^2 - 1 = \frac{1}{x^2-1}$

$y^2 = \underline{1 + \frac{1}{x^2-1}} = \frac{x^2-1+1}{x^2-1} = \underline{\frac{x^2}{x^2-1}}$

$y = \pm\sqrt{\frac{x^2}{x^2-1}}$



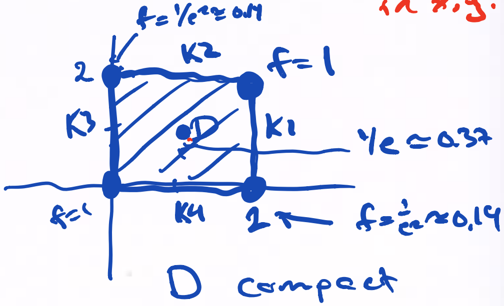
D is closed, not bounded

not compact.

5. c) max/min $f(x,y) = e^{xy-x-y}$ wh $\underline{0 \le x,y \le 2}$

symm. in x, y.

$f(x,y) = f(y,x)$



i) Interior stationary pts: $f = e^u$, $u = xy - x - y$

$f'_x = e^u \cdot (y-1) = 0$ $y = 1$

$f'_y = e^u \cdot (x-1) = 0$ $x = 1$

Candidates: $(x,y) = (1,1)$ interior pt. $f = e^{-1} = 1/e \approx 0.37$

D compact

⇓ EVT

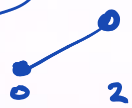
there is a max/min

ii) Other interior critical pts : no pts.

iii) Boundary pts: K1, K2, K3, K4

K1: $x = 2$, $0 \le y \le 2$ $f = f(2,y) = e^{y-2}$

$f'_y = e^{y-2} \cdot 1 > 0$



On K1:

max: $f(2,2) = 1$

min: $f(2,0) = 1/e^2 \approx 0.14$

K2: By symmetry

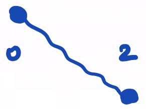
$y = 2$, $0 \le x \le 2$ $f = f(x,2) = e^{x-2}$

max $f(2,2) = 1$

min $f(0,2) = 1/e^1 \approx 0.14$

K3: $x = 0$, $0 \le y \le 2$ $f = f(0,y) = e^{-y}$

$f'_y = e^{-y} \cdot (-1) < 0$



max $f(0,0) = 1$

min $f(0,2) = 1/e^2 \approx 0.14$

K4: By symmetri

$y = 0$, $0 \le x \le 2$ $f = f(x,0) = e^{-x}$

max $f(0,0) = 1$

min $f(2,0) = 1/e^2 \approx 0.14$

Concl: max : $f = 1$ at $(0,0)$, $(2,2)$

min : $f = 1/e^2 \approx 0.14$ at $(0,2)$, $(2,0)$

Problems for Lecture 29 — EBA2910 Mathematics

Key Problems

Problem 1.
We consider the region $D \subseteq \mathbb{R}^2$ given by the inequality $y(x-2) \leq 3$. Show $D = \{(x,y) : y(x-2) \leq 3\}$ in a figure, and mark the interior points and the boundary points of D. Is D compact?

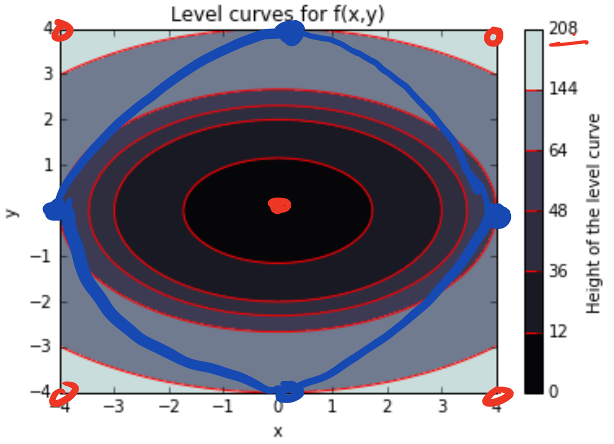
Problem 2.
We consider a subset D of the xy-plane $\mathbb{R}^2$ given by the following conditions. Determine if the subset is compact. It is useful to sketch the region D.

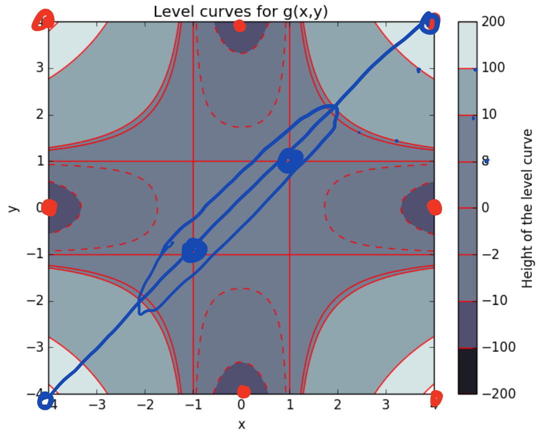
a) $2x + 3y = 6$
b) $2x + 3y < 6$
c) $2x + 3y \leq 6$
d) $x^2 + y^2 = 4$
e) $x^2 + y^2 \geq 4$
f) $x^2 + y^2 \leq 4$
g) $x^2 - 2x + 4y^2 = 4$
h) $x^2 - 2x + 4y^2 \leq 4$
i) $x^2 - 2x + 4y^2 \geq 4$
j) $xy = 1$
k) $xy \leq 1$
l) $xy \geq 1$
m) $\sqrt{x^2 + y^2} = 3$
n) $\sqrt{x^2 + y^2} \leq 3$
o) $x^2y^2 - x^2 - y^2 + 1 = 0$
p) $x^2y^2 - x^2 - y^2 + 1 = 1$

Problem 3.
What does the Extreme Value Theorem tell us? Given examples of a region D in the plane which is closed but not bounded, and a region E that is bounded but not closed. Can you find a function $f(x,y)$ that does not have a maximum and minimum in D, and a function $g(x,y)$ that does not have a maximum and minimum in E?

Problem 4.
The level curves of the functions $f(x,y) = \underline{4x^2 + 9y^2}$ and $g(x,y) = \underline{x^2y^2 - x^2 - y^2 + 1}$ in the region $-4 \leq x,y \leq 4$ are shown in the figures below:

$= (x^2-1)(y^2-1)$ ✓ $= (x^2-1)^2 \geq 0$





a) Find max / min $f(x,y)$ when $-4 \leq x,y \leq 4$ using the figure.
b) Find max / min $g(x,y)$ when $-4 \leq x,y \leq 4$ using the figure.
c) Find max / min $f(x,y)$ when $x^2 + y^2 = 16$ using the figure.
d) Find max / min $g(x,y)$ when $x = y$ using the figure.

a) $f_{max} = \underline{208}$ at $(\pm 4, \pm 4)$
$f_{min} = \underline{0}$ " $(0, 0)$

c) $f_{max} = \underline{144}$ at $(0, \pm 4)$
$f_{min} = \underline{64}$ at $(\pm 4, 0)$

b) $f_{max} = 225$ at $(\pm 4, \pm 4)$
$f_{min} = -16$ at $(\pm 4, 0), (0. \pm 4)$

d) $f_{max} = \underline{225}$ at $(\pm 4 \pm 4)$
$f_{min} = \underline{0}$ $(1,1), (-1,-1)$

Problem 5.
Solve the optimization problem:

a) $\max / \min f(x,y) = x^3 - 3xy + y^3$ when $0 \le x,y \le 1$

b) $\max / \min f(x,y) = x^3 - 3xy + y^3$ when $0 \le x,y \le 2$

c) $\max / \min f(x,y) = e^{xy-x-y}$ when $0 \le x,y \le 2$

d) $\max / \min f(x,y) = xy(x^2 - y^2)$ when $-1 \le x,y \le 1$

e) $\max / \min f(x,y) = (x^2 - 1)(y^2 - 1)$ when $-1 \le x,y \le 1$

Problem 6.
Problem 7.6.1 - 7.6.3 (norwegian textbook, optional)
Problem 9.27 - 9.31 (norwegian workbook, optional)

Answers to Key Problems

Problem 1.
Boundary points are given by the equation $y(x-2) = 3$, or points on the graph of $y = 3/(x-2)$ (a hyperbola). Interioer points are given by $y(x-2) < 3$, or points under the hyperbola when $x > 2$, and points over the hyperbola when $x < 2$, including all points with $x = 2$. The region D is not compact (it is closed but not bounded).

Problem 2.

a) No	b) No	c) No	d) Yes	e) No	f) Yes	g) Yes	h) Yes
i) No	j) No	k) No	l) No	m) Yes	n) Yes	o) No	p) No

Problem 4.

a) $f_{\text{min}} = 0$ at $(0,0)$, and $f_{\text{max}} = 208$ at $(\pm 4, \pm 4)$

b) $f_{\text{min}} = -15$ at $(0, \pm 4)$ and $(\pm 4, 0)$, and $f_{\text{max}} = 225$ at $(\pm 4, \pm 4)$

c) $f_{\text{min}} = 64$ at $(\pm 4, 0)$, and $f_{\text{max}} = 144$ at $(0, \pm 4)$

d) $f_{\text{min}} = 0$ at $(1,1)$ and $(-1,-1)$, and $f_{\text{max}} = 225$ at $(4,4)$ and $(-4,-4)$

Problem 5.

a) $f_{\text{max}} = 1,\ f_{\text{min}} = -1$

b) $f_{\text{max}} = 8,\ f_{\text{min}} = -1$

c) $f_{\text{max}} = 1,\ f_{\text{min}} = 1/e^2$

d) $f_{\text{max}} = 2\sqrt{3}/9,\ f_{\text{min}} = -2\sqrt{3}/9$

e) $f_{\text{max}} = 1,\ f_{\text{min}} = 0$