Plan

1. Method of Lagrange multipliers
2. Degenerate constraints

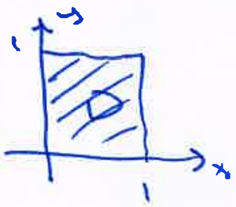
Extra lecture / problem sessions:

Lecture 32: Wed May 26 14–17
Problem Session: Thu May 27 12–15
Lecture D: Mon May 31 10–13

① Method of Lagrange multipliers

max/min $f(x,y) = \ln(xy + x + y + 1)$ when $0 \le x \le 1, 0 \le y \le 1$

D: $0 \le x, y \le 1$



D compact (closed, bounded) $\xRightarrow{\text{EVT}}$ there is a max/min

(extreme value thm.)

Candidate pts:

i) In the interior of D: stationary pts of f

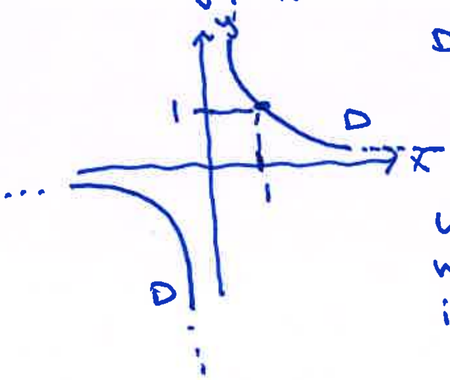
ii) Boundary pts of D

Lagrange problem: optimization problem, where all constraints are equations.

$$\boxed{\text{max/min } f(x,y) \text{ when } g(x,y) = a}$$

Ex: max/min $f(x,y) = x^2 + y^2$ when $\underbrace{xy}_{g(x,y)} = \underbrace{1}_{a}$ **Lagrange pb.**

D: $xy = 1$
$y = 1/x$



D is a curve (hyperbola)

D is not compact

We could have max/min, but it is not sure.

Substitution method:

$xy = 1 \implies y = 1/x$

$f(x,y) = x^2 + y^2$

$f(x, 1/x) = x^2 + (1/x)^2 = x^2 + 1/x^2$

$$f(x) = x^2 + 1/x^2, \quad x \ne 0$$

Ex: $x = 1$: $f(1) = 2$

$(x,y) = (1,1)$: $f(1,1) = 2$

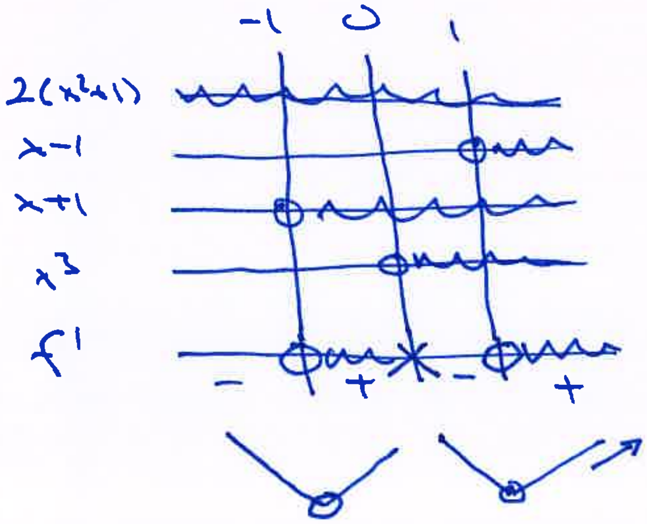
max/min $f(x) = x^2 + 1/x^2$ "x^{-2}", $x \neq 0$

$$f'(x) = 2x + (-2)x^{-3} = 2x - 2/x^3$$

$$= \frac{2x\cdot x^3}{x^3} - \frac{2}{x^3} = \frac{2(x^4-1)}{x^3}$$

$$= \frac{2(x^2-1)(x^2+1)}{x^3} = \frac{2(x^2+1)(x-1)(x+1)}{x^3}$$

Sign diagram for $f'(x)$:



$x=-1$ $x=1$

$y=1/x=-1$ $y=1/x=1$

Candidates for min:

$x=-1$: $f(-1)=2$

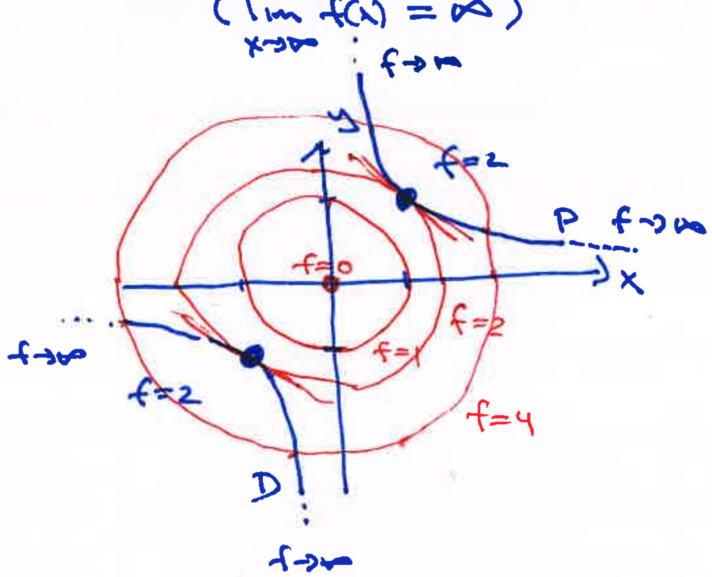
$x=1$: $f(1)=2$

$\Downarrow$

$f_{min} = 2$ at $(-1,-1)$, $(1,1)$

Max: no max

$(\lim_{x\to\infty} f(x) = \infty)$



Level curves of f:

$f(x,y)=c$

$x^2+y^2=c$

$c>0$: circle, center $(0,0)$, radius $\sqrt{c}$

$c=0$: pt. $(0,0)$

$c<0$: no pts

Method of Lagrange multipliers:

max/min $f(x,y)$ when $g(x,y) = a$

① Lagrange function: $L(x,y;\lambda) = f(x,y) - \lambda \cdot (g(x,y) - a)$
(Lagrangian)

② Find the stationary pts of L = candidates for max/min

$$\left.\begin{aligned} L'_x &= 0 \\ L'_y &= 0 \end{aligned}\right\} \text{ FOC (first order conditions)}$$

$$L'_\lambda = 0 \iff -(g(x,y) - a) = 0 \quad | \cdot (-1)$$

$$g(x,y) - a = 0$$

$$\boxed{g(x,y) = a} \quad C \text{ (constraint)}$$

Lagrange conditions: FOC+C
(conditions for stationary pts of L)

Ex: max/min $f(x,y) = x^2+y^2$ when $xy = 1$

$$L = \underbrace{x^2+y^2}_{f(x,y)} - \lambda\,(\underbrace{xy-1}_{g(x,y)-a})$$

$$\begin{array}{ll} \text{FOC} \left\{ \begin{array}{l} L'_x = \\ L'_y = \end{array} \right. & \boxed{\begin{array}{rl} 2x - \lambda(y) &= 0 \\ 2y - \lambda(x) &= 0 \\ xy &= 1 \end{array}} \\ \text{C} & \end{array}$$

FOC+C = Lagrange conditions

(1) $2x = \lambda y \Rightarrow x = \frac{\lambda y}{2}$

(2) $2y - \lambda \cdot \left(\frac{\lambda y}{2}\right) = 0 \quad | \cdot 2$

$4y - \lambda^2 y = 0$

$y(4 - \lambda^2) = 0$

$y = 0$ or $\lambda^2 = 4$

$y = 0$ or $\lambda = 2$ or $\lambda = -2$

(a) (b) (c)

(a) $\underline{y=0}$: (1) $x=0$
(3) $xy = 0 \cdot 0 \neq 1$ } no candidates in (a)

(b) $\underline{\lambda = 2}$: (1) $x = y$
(3) $xy = 1$
$y \cdot y = 1 \quad y^2 = 1 \quad \underline{y = \pm 1}$ } Candidates for (b):
$(x,y;\lambda) = \underline{(1,1;2)}, \underline{(-1,-1;2)}$
$f=2 \qquad f=2$

(c) $\underline{\lambda=-2}$: (1) $x=-y$
(3) $xy=1 \quad -y \cdot y = 1 \quad y^2 = -1$ } no candidates in c)

Result: (Necessary conditions in Lagrange problems)
If (x^*, y^*) is a max/min in a Lagrange problem, and (x^*, y^*) does not have degenerated constraint, then there is a Lagrange multiplier λ^* such that $(x^*, y^*; \lambda^*)$ satisfies the Lagrange conditions (FOC+C).

(2) Degenerated constraint

Constraint: $g(x,y) = a$

Defn: We say that the constraint is degenerated at an admissible pt (x^*, y^*) if the following condition hold:

$$\boxed{\begin{aligned} g'_x(x^*, y^*) &= 0 \\ g'_y(x^*, y^*) &= 0 \\ g(x^*, y^*) &= a \end{aligned}}$$

Ex: $xy = 1 \quad \begin{cases} g(x,y) = xy \\ a = 1 \end{cases}$

$g'_x = y = 0$
$g'_y = x = 0$ } $(x,y) = (0,0)$
$xy = 1 \qquad 0 \cdot 0 = 0 \neq 1$

} no admissible pts where the constraint is degenerate

Summary: Lagrange problem

max/min $f(x,y)$ when $g(x,y)=a$ $\Longrightarrow$

Candidate pts:

i) Pts $(x,y;\lambda)$ that satisfying the Lagrange conditions:

$L = f(x,y) - \lambda \cdot (g(x,y) - a)$

FOC: $L'_x = 0$, $L'_y = 0$

C: $g(x,y) = a$

(ordinary candidate pts)

ii) Admissible pts (x,y) with degenerate constraints: Solutions of:

$g'_x = 0$, $g'_y = 0$, $g(x,y) = a$

(exceptional candidate pts)

In Ex: i) $(x,y;\lambda) = \underset{f=2}{(1,1;2)}, \underset{f=2}{(-1,-1;2)}$

ii) no points

All candidate pts:

$\underset{f=2}{(1,1;2)}, \underset{f=2}{(-1,-1;2)}$

Part 2: Problem Set 30

1 a) $\boxed{\max/\min \ f = 3x - y \quad \text{when} \quad x^2+4y^2 = 37}$

$$\mathcal{L} = 3x - y - \lambda(x^2+4y^2-37)$$

FOC $\begin{cases} \mathcal{L}'_x = 3 - \lambda \cdot 2x = 0 \\ \mathcal{L}'_y = -1 - \lambda \cdot 8y = 0 \end{cases}$ $\Rightarrow$ $2\lambda x = 3$ $\overset{(\lambda \neq 0)}{\Rightarrow}$ $x = \frac{3}{2\lambda}$; $8\lambda y = -1 \Rightarrow y = \frac{-1}{8\lambda}$

C $\quad x^2+4y^2 = 37$

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{-1}{8\lambda}\right)^2 = 37$$

$$\frac{9}{4\lambda^2} + \frac{1}{64\lambda^2} = 37 \quad | \cdot 64\lambda^2$$

$$9 \cdot 16 + 1 = 37 \cdot 64 \lambda^2$$

$$\lambda^2 = \frac{145}{37 \cdot 64} = 0.061$$

$$\lambda = \pm\sqrt{\frac{145}{37\cdot 64}} \simeq \pm\sqrt{0.061} \simeq \pm 0.25$$

Candidate pts:

$\lambda \simeq 0.25$: $(6, -0.5; 0.25)$ $\quad f \simeq 18.5$

$\lambda \simeq -0.25$: $(-6, 0.5; -0.25)$ $\quad f \simeq -18.5$

Adm. pts with degenerate constr:

$\underbrace{x^2+4y^2=37}_{g(x,y)}$: $\quad \begin{cases} g'_x = 2x = 0 \\ g'_y = 8y = 0 \end{cases}$ $x=0$, $y=0$

$x^2+4y^2 = 37 \quad 0^2+4\cdot 0^2 \neq 37$

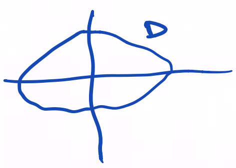
no such pts.

2a) D: $x^2+4y^2 = 37 \quad | : 37$

$$\frac{x^2}{37} + \frac{y^2}{37/4} = 1$$

ellipse, center $(0,0)$

$a = \sqrt{37}$, $b = \sqrt{37}/2$



compact

$\Downarrow$ EVT

there is a max/min

$\Rightarrow$ $f_{max} \simeq 18.5$ at $(x,y) \simeq (6, -0.5)$

$f_{min} \simeq -18.5$ " $(x,y) \simeq (-6, 0.5)$

1e) max $f = \underline{x^2y^2 - x^2 - y^2 + 16}$ when $\underline{x^2+y^2 = 16}$ i)

$= \max \ x^2y^2$ when $x^2+y^2 = 16$ ii)

ii) $\mathcal{L} = x^2y^2 - \lambda(x^2+y^2-16)$

FOC $\begin{cases} \mathcal{L}'_x = 2xy^2 - \lambda \cdot 2x = 0 \\ \mathcal{L}'_y = 2x^2y - \lambda \cdot 2y = 0 \end{cases}$ C $\{ \ x^2+y^2 = 16$

$\Rightarrow$ $2x(y^2-\lambda) = 0$, $2y(x^2-\lambda) = 0$ $\quad \begin{cases} x = 0 \text{ or } \lambda = y^2 \\ y = 0 \text{ or } \lambda = x^2 \end{cases}$

a) $x=0,\ y=0$: $0^2+0^2 \neq 16 \Rightarrow$ no sol's

b) $x=0,\ \lambda=x^2$: $\lambda=0$, $\left.\begin{array}{l} y^2=16 \\ y=\pm 4 \end{array}\right\}$ $(0,\pm 4;0)$ $f=0$

c) $\lambda=y^2,\ y=0$: $\lambda=0$ $\left.\begin{array}{l} x^2=16 \\ x=\pm 4 \end{array}\right\}$ $(\pm 4,0;0)$ $f=0$

d) $\lambda=y^2,\ \lambda=x^2$: $x^2=y^2$, $2x^2=16$, $x^2=8$ $\left.\begin{array}{l} x=\pm\sqrt{8} \\ y=\pm\sqrt{8} \end{array}\right\}$ $(\pm\sqrt{8},\pm\sqrt{8};8)$ $f=64$

Adm. pts with degenerate constr:

$\underbrace{x^2+y^2}_{g}=16$ $\left.\begin{array}{l} g'_x=2x=0 \\ g'_y=2y=0 \end{array}\right\}$ $(x,y)=(0,0)$

$x^2+y^2=16$ $0^2+0^2\neq 16$ — no such pts

2e) D: $x^2+y^2=16$ circle compact $\overset{EVT}{\Rightarrow}$ there is a max $\Rightarrow f_{max}=64$ at $(\pm\sqrt{8},\pm\sqrt{8})$

4. $g(x,y)=a$ degenerate at (x^*,y^*) if $g'_x=g'_y=0$ and $g(x,y)=a$

Ex: $x^2+y^3=0$

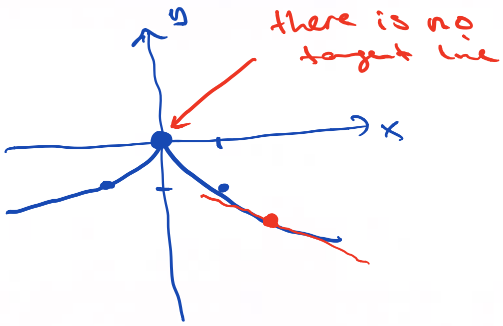
$g(x,y)=x^2+y^3$

$\left.\begin{array}{l} g'_x=2x=0 \\ g'_y=3y^2=0 \end{array}\right\}$ $(x,y)=(0,0)$

and

$g(0,0)=0^2+0^3=0$ ✓ ok

(0,0) is an adm. pt where $x^2+y^3=0$ is degenerate



$x^2+y^3=0$

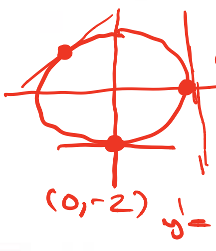
$y^3=-x^2$

$y=-\sqrt[3]{x^2}$

$x^2+y^2=4$:

$g'_x=2x$

$g'_y=2y$



(2,0) $y'=-\frac{4}{0}$

(0,−2) $y'=-\frac{0}{(-4)}=0$

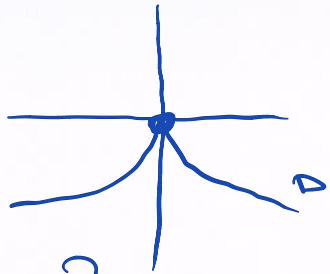
$g(x,y)=a$:

Slope of tangent line:

$$y'=-\frac{g'_x}{g'_y}$$

Ex: max $f(x,y) = y$ when $x^2+y^3=0$

$f_{max} = 0$ at $(0,0)$



Candidate pts:

i) Ordinary: $\mathcal{L} = y - \lambda(x^2+y^3)$

$$\mathcal{L}'_x = -\lambda \cdot 2x = 0 \Rightarrow \lambda = 0 \text{ or } x=0$$

$$\mathcal{L}'_y = 1 - \lambda \cdot 3y^2 = 0 \qquad 1=0 \text{ impos.} \qquad y=0,\ 1-0=0 \text{ imp.}$$

$$x^2+y^3=0$$

no ordinary cand. pt.

ii) Adm. pts with deg. constr: $(x,y) = (0,0)$ $f=0$

5. $g(x,y) = x^3+xy+y^2$ $\quad g(x,y)=0$

a) $x=-2$:

$$x^3+xy+y^2=0$$
$$(-2)^3+(-2)y+y^2=0$$
$$y^2-2y-8=0$$
$$y = \frac{2 \pm \sqrt{4-4(-8)}}{2} = \frac{2\pm 6}{2}$$
$$y=4 \quad \text{or} \quad y=-2$$
$$(-2,4) \qquad (-2,-2)$$

$$y' = -\frac{g'_x}{g'_y} = -\frac{3x^2+y}{x+2y}$$

At $(-2,4)$: $y' = -\frac{3\cdot 4+4}{-2+8} = -\frac{16}{6} = -8/3$

$$y-4 = -8/3(x+2)$$
$$y = -\frac{8}{3}x - \frac{16}{3} + 4$$
$$y = -\frac{8}{3}x - \frac{4}{3}$$

At $(-2,-2)$: $y' = -\frac{12-2}{-2-4} = +\frac{10}{6} = 5/3$

$$y+2 = 5/3(x+2)$$
$$y = \frac{5}{3}x + \frac{10}{3} - 2$$
$$y = \frac{5}{3}x + \frac{4}{3}$$

b) max $f(x,y)=x$ when $x^3+xy+y^2=0$

$$\mathcal{L}=x-\lambda(x^3+xy+y^2)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1-\lambda(3x^2+y)=0 \\ \mathcal{L}'_y &= -\lambda(x+2y)=0 \\ & x^3+xy+y^2=0 \end{aligned}}$$

$\cancel{\lambda=0}$ or $x+2y=0$

$\cancel{1-0=0}$ impossible $\quad \boxed{x=-2y}$

(3) $x^3+xy+y^2=0$

$(-2y)^3+(-2y)y+y^2=0$

$-8y^3-2y^2+y^2=0$

$-8y^3-y^2=0$

$-y^2(8y+1)=0$

$\cancel{\boxed{y=0,\ x=0}}$ or $\boxed{y=-1/8,\ x=1/4}$

(1) $x=y=0$: $1-\lambda\cdot 0=0$, $1=0$ impossible

$x=1/4,\ y=-1/8$:

$16\cdot|\quad 1-\lambda\left(3\cdot(1/4)^2-1/8\right)=0$

$16-\lambda(3-2)=0$

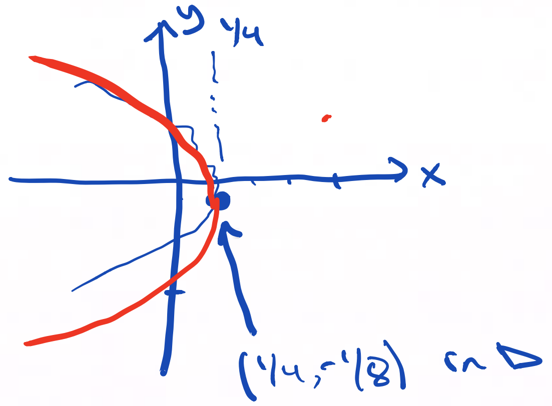
$16-\lambda=0$

$\lambda=16$

One ordinary candidate pt: $(x,y;\lambda)=(1/4,-1/8;16)$

$f=1/4$

Must check if $f=1/4$ is the maximum value:



D: $x^3+xy+y^2=0$

For what values of x are there solutions for y in

$x^3+xy+y^2=0$

$x=1$: $1+y+y^2=0$

$y=\dfrac{-1\pm\sqrt{1-4}}{2}$ not possible

$x=a$: $a^3+ay+y^2=0$

$y^2+ay+a^3=0$

$y=\dfrac{-a\pm\sqrt{a^2-4\cdot1\cdot(a^3)}}{2}$

$\Delta=a^2-4a^3$

$=a^2(1-4a)$

1/4

a^2

$1-4a$

Δ

$a>1/4$: no solutions

$a\le 1/4$: solutions

Conclusion:

$f_{max}=1/4$ at $(1/4,-1/8)$

7. max $f(x,y) = x+y$ when $x^3 - 3xy + y^3 = 0$ (not included in video)

$$\mathcal{L} = x + y - \lambda(x^3 - 3xy + y^3)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1 - \lambda \cdot (3x^2 - 3y) = 0 \\ \mathcal{L}'_y &= 1 - \lambda(-3x + 3y^2) = 0 \\ & x^3 - 3xy + y^3 = 0 \end{aligned}}$$

$$\left.\begin{aligned} 3x^2 - 3y &= 1/\lambda \\ 3y^2 - 3x &= 1/\lambda \end{aligned}\right\} (\lambda \neq 0)$$

$$\parallel$$

$$3x^2 - 3y = 3y^2 - 3x$$

$$3x^2 - 3y^2 = 3y - 3x \quad 1:3$$

$$x^2 - y^2 = y - x$$

$$(x-y)(x+y) = -1 \cdot (x-y)$$

$$(x-y)(x+y) + 1 \cdot (x-y) = 0$$

$$(x-y)(x+y+1) = 0$$

$x = y$ or $x + y + 1 = 0$

a) $x = y$: $x^3 - 3x^2 + x^3 = 0$

$$2x^3 - 3x^2 = 0$$

$$x^2(2x - 3) = 0$$

$x = 0$ or $x = 3/2$

$y = 0$ $\quad$ $y = 3/2$

(1) $1 - \lambda \cdot 0 = 0$ impossible

(1) $1 - \lambda \cdot (3 \cdot 9/4 - 3/2 \cdot 3) = 0$

$$1 - \lambda\left(\frac{27}{4} - \frac{9 \cdot 2}{2 \cdot 2}\right)$$

$$\frac{9}{4}\lambda = 1$$

$$\lambda = 4/9$$

Cand. pt: $(3/2, 3/2; 4/9)$

$f = 3$

b) $x + y + 1 = 0$: $x = -(y+1)$

$$x^3 - 3xy + y^3 = 0$$

$$[-(y+1)]^3 - 3 \cdot [-(y+1)] \cdot y + y^3 = 0$$

$$-(y^3 + 3y^2 + 3y + 1) + 3(y+1)y + y^3 = 0$$

$$-\cancel{y^3} - \cancel{3y^2} - \cancel{3y} - 1 + \cancel{3y^2} + \cancel{3y} + \cancel{y^3} = 0$$

$$1 = 0$$

impossible

Conclusion:

Since we assume there is a max, it must be $f_{max} = 3$ at $(3/2, 3/2)$

(there is one adm. pt. with degenerate constr., see next page, but it has smaller f-value)

Adm pts with degenerate constraints:

$x^3 - 3xy + y^3 = 0$

$g'_x = 3x^2 - 3y = 0$
$g'_y = -3x + 3y^2 = 0$
$x^3 - 3xy + y^3 = 0$

$\left.\begin{matrix} y = x^2 \\ x = y^2 \end{matrix}\right\} \Rightarrow x = y^2 = (x^2)^2 = x^4$

$x - x^4 = 0$

$x(1 - x^3) = 0$

$\underline{x=0}$ or $\underline{x=1}$

$y = 0 \qquad y = 1$

$0^3 - 3 \cdot 0 + 0^3 = 0 \qquad 1^3 - 3 \cdot 1 \cdot 1 + 1^3 \neq 0$

ok $\qquad$ not adm

one cand. pt: $(x,y) = (0,0)$

$\underline{f = 0}$