Plan

1. Explanation: Method of Lagrange multipliers
2. Interpretation of the Lagrange multipliers

ELE3781: Elective mathematics
Maths + Python

Ex: max $f(x,y) = x^2y^2$ when $x^2+y^2+x^2y^2 = 3$

$D: x^2+y^2+x^2y^2 = 3$

D is compact (closed and bounded)

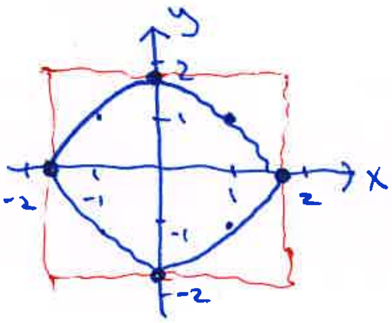
$x^2 \le 3 \quad y^2 \le 3$

$\Updownarrow \qquad \Updownarrow$

$-\sqrt{3} \le x \le \sqrt{3} \quad -\sqrt{3} \le y \le \sqrt{3}$

$\Downarrow$ EVT

there is a max



$D: x^2+y^2+x^2y^2 = 3$

$$y^2 + x^2y^2 = 3-x^2$$

$$y^2(1+x^2) = 3-x^2$$

$$y^2 = \frac{3-x^2}{1+x^2}$$

$$y = \pm\sqrt{\frac{3-x^2}{1+x^2}}$$

① Method of Lagrange multipliers

max $f(x,y) = x^2y^2$ when $\underbrace{x^2+y^2+x^2y^2}_{g(x,y)} = \underbrace{3}_{a}$

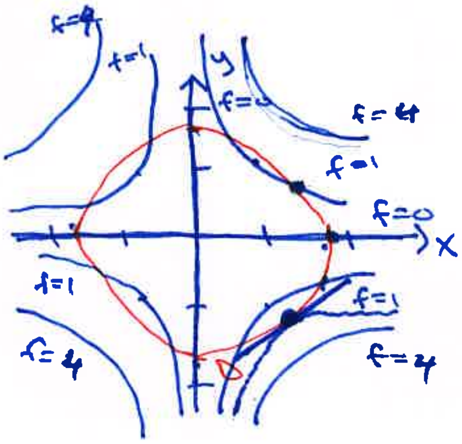
$$L = f(x,y) - \lambda \cdot (g(x,y) - a)$$
$$= x^2y^2 - \lambda \cdot (x^2+y^2+x^2y^2-3)$$

$$L'_x = f'_x - \lambda \cdot g'_x = 2xy^2 - \lambda \cdot (2x + 2xy^2) = 0$$
$$L'_y = f'_y - \lambda \cdot g'_y = 2x^2y - \lambda(2y + 2x^2y) = 0$$

} FOC

$$x^2 + y^2 + x^2y^2 = 3$$

} C



D: $g(x,y) = a$
$x^2+y^2+x^2y^2 = 3$

Level curves for f: $f(x,y) = c$
$x^2y^2 = c$

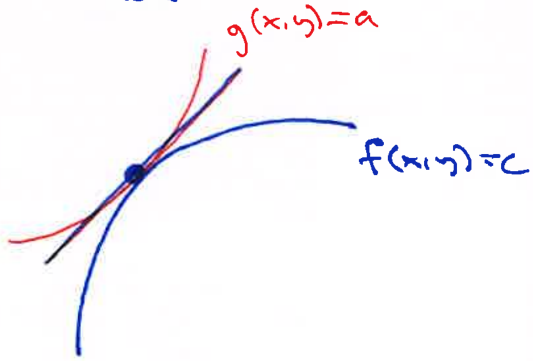
$c > 0$	$c = 0$	$c < 0$
$x^2y^2 = c$	$x^2y^2 = 0$	no points
$y^2 = c/x^2$	$x = 0$ or $y = 0$	
$y = \pm\sqrt{c/x^2}$	$\Downarrow$	
$= \pm\sqrt{c}/x$	x-axis and y-axis	
$y = \frac{\sqrt{c}}{x}$ or $y = -\frac{\sqrt{c}}{x}$		

Ex:
$c=1$: $y = 1/x$ or $y = -1/x$
$c=4$: $y = 2/x$ or $y = -2/x$

Fact: If (x^*, y^*) is a max then the set D (red curve) and the level curve of f at (x^*, y^*) (blue curve) meet at a tangent.

That is: The tangent of $g(x,y) = a$ at (x^*, y^*)
= the tangent of the level curve of f at (x^*, y^*)

To find candidate pts we therefore solve the equation



$$\underbrace{-\frac{g'_x}{g'_y}}_{\substack{\text{slope of tangent}\\ \text{(red curve)}}} = \underbrace{-\frac{f'_x}{f'_y}}_{\substack{\text{slope of tangent}\\ \text{(blue curve)}}}$$

$$-\frac{g'_x}{g'_y} = -\frac{f'_x}{f'_y} \quad \Big| \cdot (-1) \cdot g'_y \cdot f'_y$$

$$g'_x f'_y = f'_x g'_y \quad \Big| : (g'_x \cdot g'_y)$$

$$\frac{\cancel{g'_x} f'_y}{\cancel{g'_x} g'_y} = \frac{f'_x \cancel{g'_y}}{g'_x \cancel{g'_y}}$$

$$\frac{f'_y}{g'_y} = \frac{f'_x}{g'_x} =: \lambda$$

call the common quotient $\underline{\lambda}$ = Lagrange multiplier

$$\frac{f'_x}{g'_x} = \lambda \quad \text{and} \quad \frac{f'_y}{g'_y} = \lambda$$

$$\Updownarrow$$

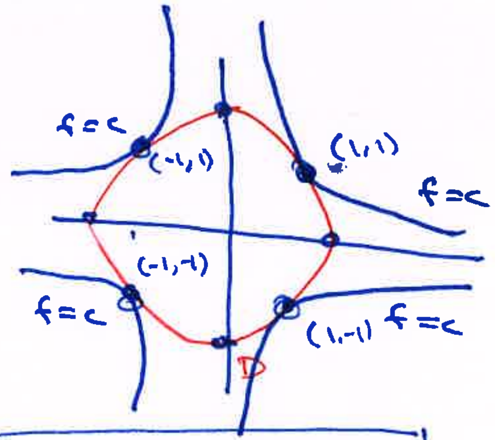
$$f'_x = \lambda g'_x \qquad f'_y = \lambda g'_y$$

$$\boxed{L'_x = f'_x - \lambda g'_x = 0 \qquad L'_y = f'_y - \lambda g'_y = 0}$$ FOC

max $f(x,y)=x^2y^2$ when $x^2+y^2+x^2y^2=3$

$$L = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$$

$$\boxed{\begin{aligned} L'_x &= 2xy^2 - \lambda(2x+2xy^2) = 0 \\ L'_y &= 2x^2y - \lambda(2y+2x^2y) = 0 \\ & x^2+y^2+x^2y^2 = 3 \end{aligned}}$$



(1) $2x(y^2-\lambda(1+y^2))=0 \iff$ $x=0$ or $y^2=\lambda(1+y^2)$

(2) $2y(x^2-\lambda(1+x^2))=0$ $\quad$ $y=0$ or $x^2=\lambda\cdot(1+x^2)$

a) $x=0$, $y=0$	b) $x=0$, $x^2=\lambda(1+x^2)$	c) $y^2=\lambda(1+y^2)$, $y=0$	d) $y^2=\lambda(1+y^2)$ ✓, $x^2=\lambda(1+x^2)$ ✓
$x=0,\ y=0$	$x=0,\ \lambda=0$	$y=0,\ \lambda=0$	$\lambda = \boxed{\dfrac{y^2}{1+y^2} = \dfrac{x^2}{1+x^2}} = \lambda$
$0+0+0=3$	$0+y^2+0=3$	$x^2+0+0=3$	$y^2(1+x^2)=x^2(1+y^2)$
$\Downarrow$	$y=\pm\sqrt{3}$	$x=\pm\sqrt{3}$	$y^2+\cancel{x^2y^2}=x^2+\cancel{x^2y^2}$
no pts	$\Downarrow$	$\Downarrow$	$y^2=x^2$
	$(0,\pm\sqrt{3};0)$ $f=0$	$(\pm\sqrt{3},0;0)$ $f=0$	$x^2+x^2+x^2\cdot x^2=3$
			$x^4+2x^2-3=0$
			$(x^2+3)(x^2-1)=0$
			$\cancel{x^2=-3}$ or $x^2=1$
			$x=\pm1$
			$y=\pm1$
			$\Downarrow$ $\lambda=1/2$
			$(\pm1,\pm1;1/2)$ $f=1$

Check for exceptional candidate pts:

Adm pts where the constr. is degenerate

$\underbrace{x^2+y^2+x^2y^2}_{g(x,y)}=3$

$$\begin{cases} g'_x = 2x+2xy^2=0 \\ g'_y = 2y+2x^2y=0 \\ x^2+y^2+x^2y^2=3 \end{cases}$$

$2x(1+y^2)=0 \Rightarrow x=0$

$2y(1+x^2)=0 \Rightarrow y=0$

$0+0+0\neq3$

no adm pts with deg. constr.

Conclusion: $f_{max}=1$ at $(\pm1,\pm1)$ with $\lambda=1/2$ Lagrange multiplier

② Interpretation of Lagrange multipliers : λ

Ex: $\boxed{\max\ f = x^2y^2 \text{ when } x^2+y^2+x^2y^2 = 3}$ $\Rightarrow f_{max} = 1$ at $(\pm1, \pm1)$ with $\lambda = 1/2$

$\max\ f = x^2y^2$ when $x^2+y^2+x^2y^2 = a$

Lagrange pb. with parameter a

Write : $f^*(a)$ for the maximum value

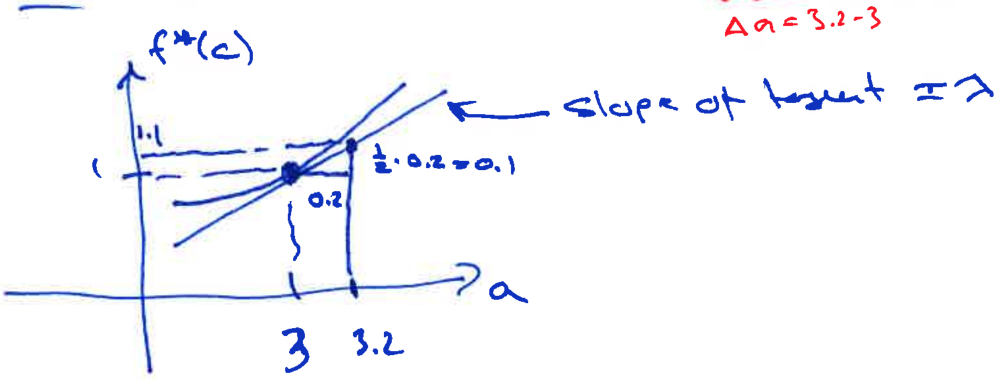
Ex: $f^*(3) = 1$

Interpretation: $\dfrac{df^*(a)}{da} = \lambda$ marginal change in max. value when we change a, the constant in the constraint

slope $= \lambda$

Ex: $f^*(3.2) \simeq f^*(3) + 0.2 \cdot \frac{1}{2} = 1 + 0.1 = 1.1$

$\Delta a = 3.2 - 3$



Part 2: Problem Set 31

5. $C: \; y(x^2+y^2) = 2(x^2-y^2)$

a) $y=-1$: $-1\cdot(x^2+1) = 2(x^2-1)$

$$-x^2-1 = 2x^2-2$$

$$\frac{-3x^2}{-3} = \frac{-1}{-3} \qquad x^2 = 1/3$$

$$x = \pm\sqrt{1/3} = \pm\frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

Pts: $(\pm\sqrt{1/3}, -1)$

b) $y(x^2+y^2) = 2(x^2-y^2) \iff \underbrace{y(x^2+y^2)-2(x^2-y^2)}_{g(x,y)} = \underset{a}{0}$

$$y' = -\frac{g'_x}{g'_y} = -\frac{2x(y-2)}{x^2+3y^2+4y}$$

$$g'_x = y(2x)-2(2x) = 2x(y-2)$$

$$g'_y = x^2+3y^2+4y$$

$(\sqrt{1/3}, -1)$: $y' = -\frac{2\sqrt{1/3}\,(-3)}{1/3-1} = \frac{6\sqrt{1/3}}{-2/3} = -9\sqrt{1/3}$

$$= -9\cdot\sqrt{3}/3 = -3\sqrt{3}$$

$$y+1 = -3\sqrt{3}(x-\sqrt{1/3}) = -3\sqrt{3}\,x+3$$

$$y = -3\sqrt{3}x+2$$

$(-\sqrt{1/3}, -1)$: $y' = -\frac{2(-\sqrt{1/3})(-3)}{1/3-1} = 3\sqrt{3}$

$$y+1 = 3\sqrt{3}(x+\sqrt{1/3}) = 3\sqrt{3}\,x+3$$

$$y = 3\sqrt{3}x+2$$

c) max/min $f(x,y) = y$ when $\underbrace{y(x^2+y^2) - 2(x^2-y^2)}_{g(x,y)} = \underbrace{0}_{a}$

i) Lagrange: $L = y - \lambda\left(y(x^2+y^2) - 2(x^2-y^2)\right)$

$$\boxed{\begin{aligned} L'_x &= -\lambda\left(2x(y-2)\right) = 0 \\ L'_y &= 1 - \lambda\cdot\left(x^2+3y^2+4y\right) = 0 \\ & y(x^2+y^2) - 2(x^2-y^2) = 0 \end{aligned}}$$

i) $\lambda = 0$ or $2x = 0$ or $y - 2 = 0$

ii) $1 - 0 = 0$ ⇓ no pts

$x = 0$: iii) $y \cdot y^2 - 2(-2y^2) = 0$, $y^3 + 2y^2 = 0$, $y^2(y+2) = 0$

$y = 0$: $1 - \lambda \cdot 0 = 0$, $1 = 0$ ⇓ no pts

$y = -2$: $1 - \lambda(4) = 0$, $\lambda = 1/4$

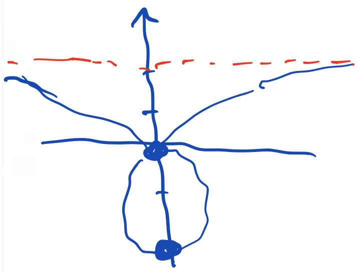
$(0,-2;1/4)$ $f = -2$

not easy to determine if this is max/min

$y = 2$: iii) $2(\cancel{x^2}+4) - \cancel{2(x^2}-4) = 0$, $8 + 8 = 0$, $16 = 0$ ⇓ no pts

ii) Alternative method: max/min y when $y(x^2+y^2) = 2(x^2-y^2)$

For which values of y does the eqn have solution for x? $[-2, 2)$



$y = a$: $a(x^2+a^2) = 2(x^2-a^2)$

$ax^2 + a^3 = 2x^2 - 2a^2$

$ax^2 - 2x^2 = -a^3 - 2a^2$

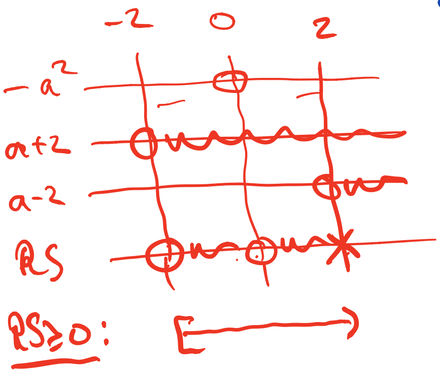
$(a-2)x^2 = -a^3 - 2a^2 = \dfrac{-a^2(a+2)}{a-2}$

$\underline{a-2}$

$x^2 = \dfrac{-a^2(a+2)}{a-2}$ ← look at sign of

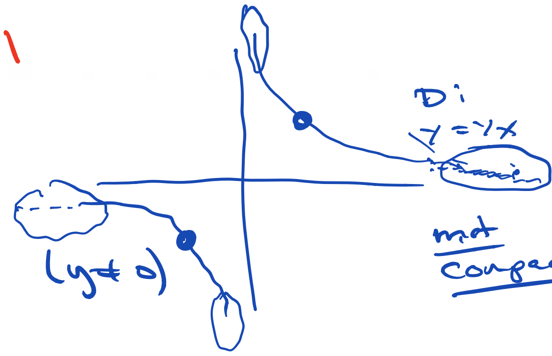
$f_{min} = -2$ at $(0,-2)$

no max



6. max/min $f = x^3 + 3xy + y^3$ when $xy = 1$

$= \max/\min\ x^3 + 3 + y^3$ when $xy = 1$



not compact

$$\mathcal{L} = x^3 + 3 + y^3 - \lambda(xy - 1)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 3x^2 - \lambda y = 0 \\ \mathcal{L}'_y &= 3y^2 - \lambda x = 0 \\ xy &= 1 \end{aligned}}$$

$\Rightarrow \lambda = \frac{3x^2}{y}$ $(y \neq 0)$

$\Rightarrow \lambda = \frac{3y^2}{x}$ $(x \neq 0)$

$$\frac{3x^2}{y} = \frac{3y^2}{x} \quad | \cdot xy$$

$$3x^2 \cdot x = 3y^2 \cdot y \quad | : 3$$

$$x^3 = y^3 \quad | \sqrt[3]{\ }$$

$$x = y$$

no adm. pts on $xy = 1$ with degenerate constr.

$$y \cdot y = 1$$

$$y^2 = 1$$

$$y = \pm 1$$

$$\Downarrow$$

$(1, 1; 3)$ or $(-1, -1; -3)$

$f = 5$ $\quad$ $f = 1$

max? **NO** $\quad$ min? **NO**

$f(4, 1/4) = 4^3 + 3 + (1/4)^3 > 67$

$f(-4, -1/4) = (-4)^3 + 3 + (-1/4)^3 = -64 + 3 - (1/4)^3 < -61$

$f(x, 1/x) \to \infty$ as $x \to \infty$

$\Rightarrow$ there is no max or min