Plan

1 About the exam
2 Exam EBA2910 06/2020

Reminder: i) Course evaluation

ii) Lecture D: Monday

Problem session: Thursday

Important: Write what you want me to do on Monday. (in Padlet)

① About the exam:

June 2nd, 10.00 – 15.15: Exam time 5h (+ 15 min)

$\underbrace{\text{15 questions}}_{15 \times 6p = 90p} + \underbrace{\text{extra credit questions}}_{\text{can be skipped}}$

Requirement to pass: 40% – 60%

Questions: $\boxed{\begin{array}{l}\text{All earlier exams} \\ \text{Course papers (EBA 29103)}\end{array}}$

– integration
– matrix / vector
– fn. in two variables
– fn. in one variable (derivation)

2 Exam EBA 2910 06/2020

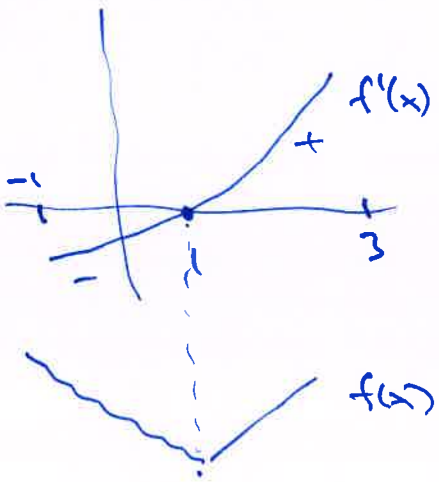
1. a) Slope of tangent in $x=1$:

i) $f'(1) = \underline{0}$

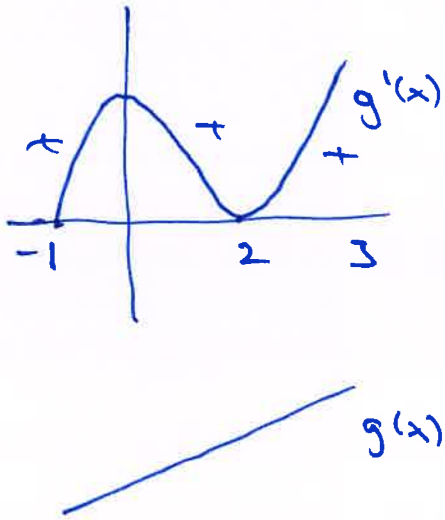
ii) $g'(1) = \underline{2}$

read off values of the derivative from the figure

b)

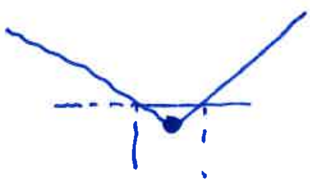


$x=1$ is min of f



$x=-1$ is min for g

c) f has no inverse



not uniquely defined since f has a min in the interior

g has an inverse

since g is increasing function

2. $A = \begin{pmatrix} 3 & a & -2 \\ a & a^2+1 & -a \\ -2 & -a & 3 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $A \cdot \underline{x} = \underline{b}$

a) $a = 2$:

$$\left(\begin{array}{ccc|c} 3 & 2 & -2 & 1 \\ 2 & 5 & -2 & 0 \\ -2 & -2 & 3 & 1 \end{array}\right) \cdot (-1) \to \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 2 & 5 & -2 & 0 \\ -2 & -2 & 3 & 1 \end{array}\right) \begin{matrix} -2 \\ 2 \end{matrix}$$

extended matrix

$$\to \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 11 & -2 & -2 \\ 0 & -8 & 3 & 3 \end{array}\right) \cdot 8/11 \to \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 11 & -2 & -2 \\ 0 & 0 & 3-\frac{16}{11} & 3-\frac{16}{11} \end{array}\right) \cdot 11$$

$$\to \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 11 & -2 & -2 \\ 0 & 0 & 17 & 17 \end{array}\right)$$

echelon form

$$\begin{aligned} x - 3y &= 1 \\ 11y - 2z &= -2 \\ 17z &= 17 \end{aligned}$$

$z = 1$

$11y - 2 = -2$ $\quad y = 0$

$x - 3 \cdot 0 = 1$ $\quad x = 1$

One solution: $(x, y, z) = (1, 0, 1)$

b) $|A| = \begin{vmatrix} 3 & a & -2 \\ a & a^2+1 & -a \\ -2 & -a & 3 \end{vmatrix} = 3 \cdot \left(3(a^2+1) - a^2\right) - a\left(3a - 2a\right) + (-2) \cdot \left(-a^2 + 2(a^2+1)\right)$

$= 3(2a^2+3) - a^2 - 2(a^2+2) = 3a^2 + 5$

$A\underline{x} = \underline{b}$ exactly one solution $\Leftrightarrow |A| \neq 0$

$|A| = 0$: $3a^2 + 5 = 0$ $\Rightarrow$ Exactly one solution for all values of a

$a^2 = -5/3$

no solutions

c) $\underline{a=0}$: Compute A^{-1}.

$$A^{-1} = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T}_{\mathrm{adj}(A)}$$

$$= \frac{1}{5} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}^T$$

$$= \underline{\underline{\frac{1}{5} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}}}$$

$\underline{a=0}$:

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{pmatrix}$$

$$|A| = 3 \cdot 0^2 + 5$$
$$= \underline{\underline{5}} \quad \text{from b)}$$

$C_{11} = 3$ $C_{12} = 0$ $C_{13} = 2$
$C_{21} = 0$ 5 0
2 0 3

d) $A^n \cdot \underline{b}$ when $a=2$, n large integer

$$A \cdot \underline{b} = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \underline{b}$$

$$A^2 \cdot \underline{b} = A \cdot (A \cdot \underline{b}) = A \cdot \underline{b} = \underline{b}$$

$\vdots$

$$A^n \cdot \underline{b} = A^{n-1} \cdot (A \cdot \underline{b}) = A^{n-1} \cdot \underline{b} = A^{n-2} \cdot \underline{b} = \ldots = A \cdot \underline{b} = \underline{b}$$

$\Vert$

$$A^n \cdot \underline{b} = b = \underline{\underline{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}} \quad \text{for all large integers } n \geq 1$$

3. $f(x) = \dfrac{Q(x)}{L(x)}$ $\begin{matrix} \leftarrow \text{deg. } 2 \\ \leftarrow \text{deg. } 1 \end{matrix}$

Ex: $\dfrac{x^2}{x-1} = \underline{x+1} + \dfrac{1}{x-1}$

$= \underbrace{a(x)}_{\text{deg } I} + \dfrac{b}{L(x)}$ $\leftarrow$ konst.

a) From this we find:

$L(x)=0$: vertical asymptote $x=2$

$y=a(x)$: skew asymptote $y=-x-2$

b) $f(x) = -x-2 + \dfrac{b}{x-2}$

Read off: $(x,y) = (1,-2)$

$-2 = -1-2 + \dfrac{b}{1-2}$

$-2 = -3 - b$

$-2+3 = -b$

$1 = -b \qquad b = -1$

$f(x) = -x-2 + \dfrac{-1}{x-2}$

$= \dfrac{(-x-2)(x-2) - 1}{x-2} = \dfrac{-x^2+4-1}{x-2} = \dfrac{3-x^2}{x-2}$

$f'(x) = -1 - 1\cdot(-1)(x-2)^{-2}\cdot 1 = -1 + \dfrac{1}{(x-2)^2}$

c) Global max/min: $\Rightarrow$ stationary pts

$f'(x) = 0 : -1 + \dfrac{1}{(x-2)^2} = 0$

$\dfrac{1}{(x-2)^2} = 1$

$1 = (x-2)^2$

$\pm 1 = \pm\sqrt{1} = x-2$

$x = 2 \pm 1 = 3, 1$

See from figure that $x=1,3$ are local max/min but not global max/min

$\Downarrow$

no global max/min

4. a) $\int x(1-x)^2\,dx = \int x(1-2x+x^2)\,dx$

$= \int x - 2x^2 + x^3\,dx = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + C$

power rule

b) $\int \frac{x}{1-x^2}\,dx = \int \frac{\cancel{x}}{u}\cdot\frac{du}{-2\cancel{x}} = -\frac{1}{2}\int \frac{1}{u}\,du$

$\boxed{\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array}}$

substitution

$= -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|1-x^2| + C$

c) $\int \frac{x}{(1-\sqrt{x})^2}\,dx = \int \frac{x}{u^2}(-2\sqrt{x})\,du$

$\boxed{\begin{array}{l} u = 1-\sqrt{x} \\ du = -\frac{1}{2\sqrt{x}}\,dx \end{array}}$

substitution

$\sqrt{x} = 1-u$

$= \int \frac{-2x\sqrt{x}}{u^2}\,du = \int \frac{-2(1-u)^2(1-u)}{u^2}\,du$

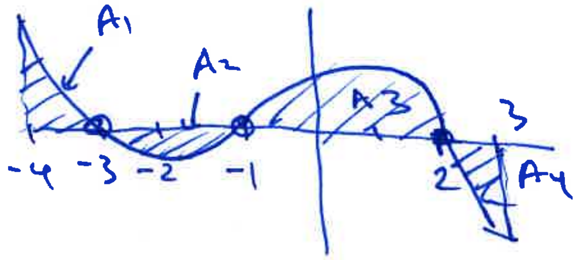
$= -2\int \frac{(1-2u+u^2)(1-u)}{u^2}\,du = -2\int \frac{1-3u+3u^2-u^3}{u^2}\,du$

$= -2\int \frac{1}{u^2} - \frac{3}{u} + 3 - u\,du$

$= -2\left(-\frac{1}{u} - 3\ln|u| + 3u - \frac{1}{2}u^2\right) + C$

$= \frac{2}{1-\sqrt{x}} + 6\ln|1-\sqrt{x}| - 6(1-\sqrt{x}) + (1-\sqrt{x})^2 + C$

d) $A(a)=\int_{-4}^{a} f(x)\,dx$: For which value of a is this maximal



$a=-3$: A_1

$a=-1$: $A_1 - A_2$

$a=2$: $A_1 - A_2 + A_3$

$A(-1) < A(-3)$

$A(2) > A(-3)$

$\underline{\underline{a=2}}$ is maximum pt for A

Candidates for max:

- stationary pts
  $A'(a)=0 \iff f(a)=0$ intersection with x-axis

- boundary pts : $a=-4$, $a=3$

$A(-4)=0 < A(2)$

$A(3) = \underline{A_1 - A_2 + A_3} - A_4 < A(2)$

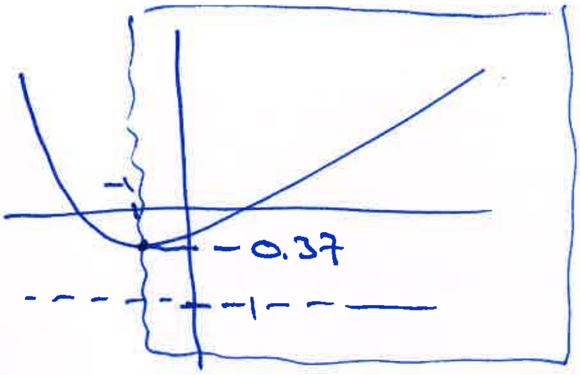
$\underline{\underline{a=2}}$ is max since $[-4,3]$ is compact

$\Downarrow$ EVT

there is a max

5. $f(x) = xe^x$

a) Explain why $f(x) = -1$ has no solutions



$f'(x) = 1 \cdot e^x + x \cdot e^x = (x+1)e^x$

		-1	
$x+1$	$-$	0	$+$
e^x	$+$		$+$
$f'(x)$	$-$	0	$+$

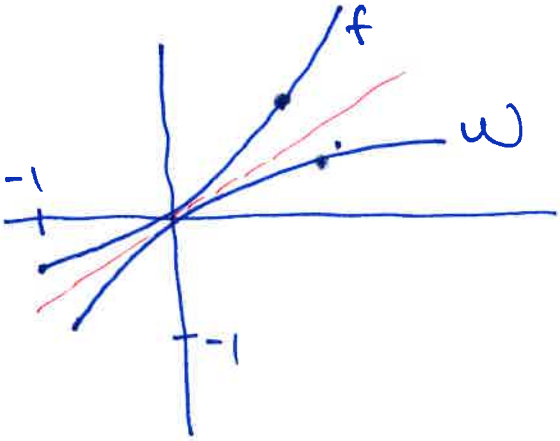
Since $f_{min} \simeq -0.37 > -1$
$f(x) = -1$ has no solutions

$x=-1$ is global min

$f(-1) = -1 \cdot e^{-1} = -1/e$
$\simeq -0.37$
min. value

b) $f(x) = xe^x$, $x \geq -1$

$W = f^{-1}$



Graph of f^{-1} = mirror image of the graph of f along $y=x$

$\Downarrow$

W increasing

$W(f(x)) = x$

$W'(f(x)) \cdot f'(x) = 1$

$W'(f(x)) = 1/f'(x) > 0$

since $f'(x) > 0$

$\Downarrow$

W increasing fn

6. $C: \; 4x^2 - 24x + t^2y^2 = 64$ (t parameter)

a) $\underline{t \neq 0}$: C is an ellipse

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

$x_0 = 3$

$y_0 = 0$

$a^2 = \frac{100}{4} = 25$

$b^2 = \frac{100}{t^2}$

$$4x^2 - 24x + t^2y^2 = 64$$

$$4(x^2 - 6x + 9) + t^2(y^2) = 64 + 36$$

$$4(x-3)^2 + t^2 \cdot y^2 = 100$$

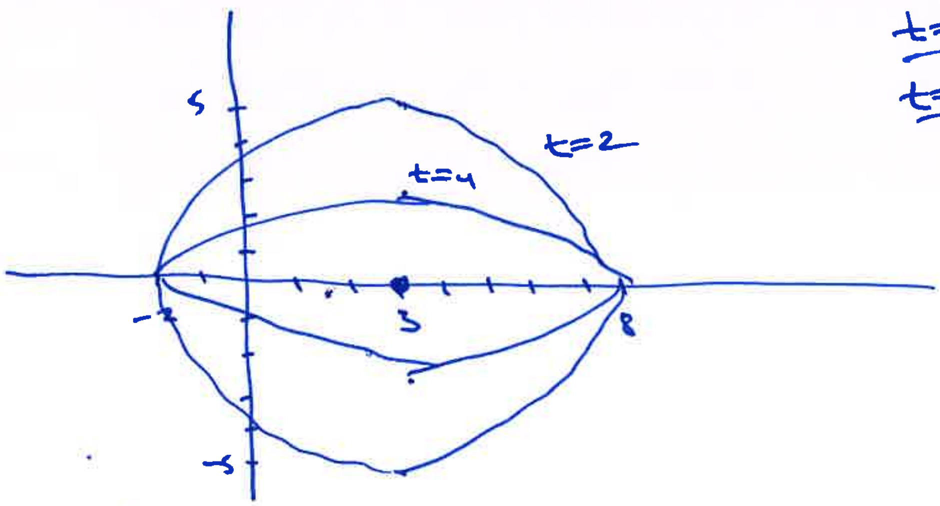
$$\frac{4(x-3)^2}{100} + \frac{t^2y^2}{100} = 1$$

Ellipse, center $(3,0)$!

half-axis $a = 5$, $b = 10/|t|$

$t=2$: $b = 5$

$t=4$: $b = 5/2 = 2.5$



b) $f(x,y) = xy$

$f'_x = y = 0$

$f'_y = x = 0$

Stat. pts: $(x,y) = (0,0)$

$$H(f) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det = 0 - 1 = -1 < 0$

$\Downarrow$ second derivative test

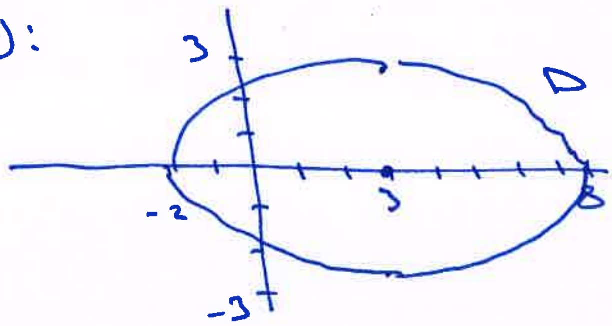
$(0,0)$ is saddle pt

e) max $f(x,y) = xy$ when $4x^2 - 24x + 16y^2 = 64$
(C for $t=4$)

Note: i) $D = \{(x,y) : 4x^2 - 24x + 16y^2 = 64\}$
set of adm. pts.
is an ellipse (C for $t=4$)

D is bounded $\underset{EVT}{\Rightarrow}$ there is a max

ii) From a):



No adm. pts with degenerate constraints.
($g'_x = g'_y = 0$)

$$L = xy - \lambda(4x^2 - 24x + 16y^2 - 64)$$

$$\boxed{\begin{aligned} L'_x &= y - \lambda(8x-24) = 0 \\ L'_y &= x - \lambda(32y) = 0 \\ & 4x^2 - 24x + 16y^2 = 64 \end{aligned}}$$

Lagrange conditions
Solution $\Rightarrow$ cand. pts

(1) $\cancel{y = \lambda(8x-24)}$
(2) $\cancel{x - \lambda \cdot 32 \cdot \lambda(8x-24) = 0}$

(1) $\lambda = \dfrac{y}{8x-24}$
(2) $\lambda = \dfrac{x}{32y}$
$\|$

$$\frac{y}{8x-24} = \frac{x}{32y}$$

$32y^2 = x \cdot (8x-24) \quad :2$
$16y^2 = x(4x-12)$

Check: $8x - 24 = 0$: $x=3, y=0, \lambda=0$
impossible

$32y = 0$: $y=0, x=0, \lambda=0$
$0 \neq 64$
impossible

(3) $4x^2 - 24x + (4x^2 - 12x) = 64$

$16y^2 = 4x^2 - 12x$

$\boxed{y^2 = \frac{4x^2 - 12x}{16}}$

$8x^2 - 36x - 64 = 0 \quad 1:4$

$2x^2 - 9x - 16 = 0$

$$x = \frac{9 \pm \sqrt{81 - 4 \cdot 2 \cdot (-16)}}{2 \cdot 2} = \frac{9 \pm \sqrt{209}}{4}$$

$\underline{x_1 \simeq 5.864}$, $\underline{x_2 = -1.364}$

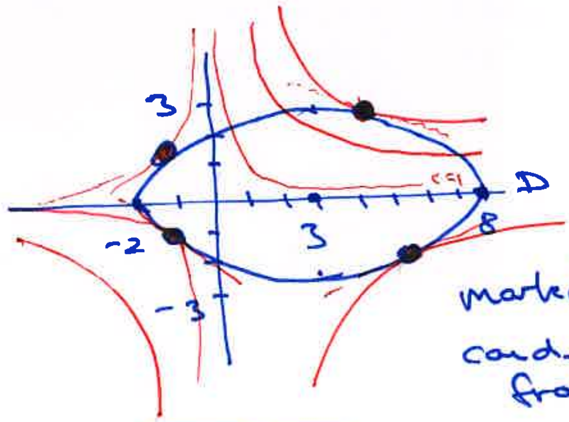
$\underline{y_1 \simeq \pm 2.049}$ $\quad \underline{y_2 = \pm 1.220}$.

four candidate pts

Compute $f(x,y) = xy$ and compare

$$f_{max} \simeq f(5.864, 2.049) \simeq \underline{\underline{12.02}}$$

d) $4x^2 - 24x + 16y^2 = 64$
(ellipse)



marked pts = cand. pts from c)

<u>Level curves for f:</u>

$f(x,y) = c$

$xy = c$

$y = c/x$

<u>From theory:</u>

candidate pts = adm pts where the level curve of f meets the ellipse at a tangent