EBA 2911, lecture 4, 2 Sept. 2020, Runar Ile

Plan:
1. Regular cash flows
2. Infinite series and limit values
3. Euler's number and continuous compounding

1. Regular cash flows

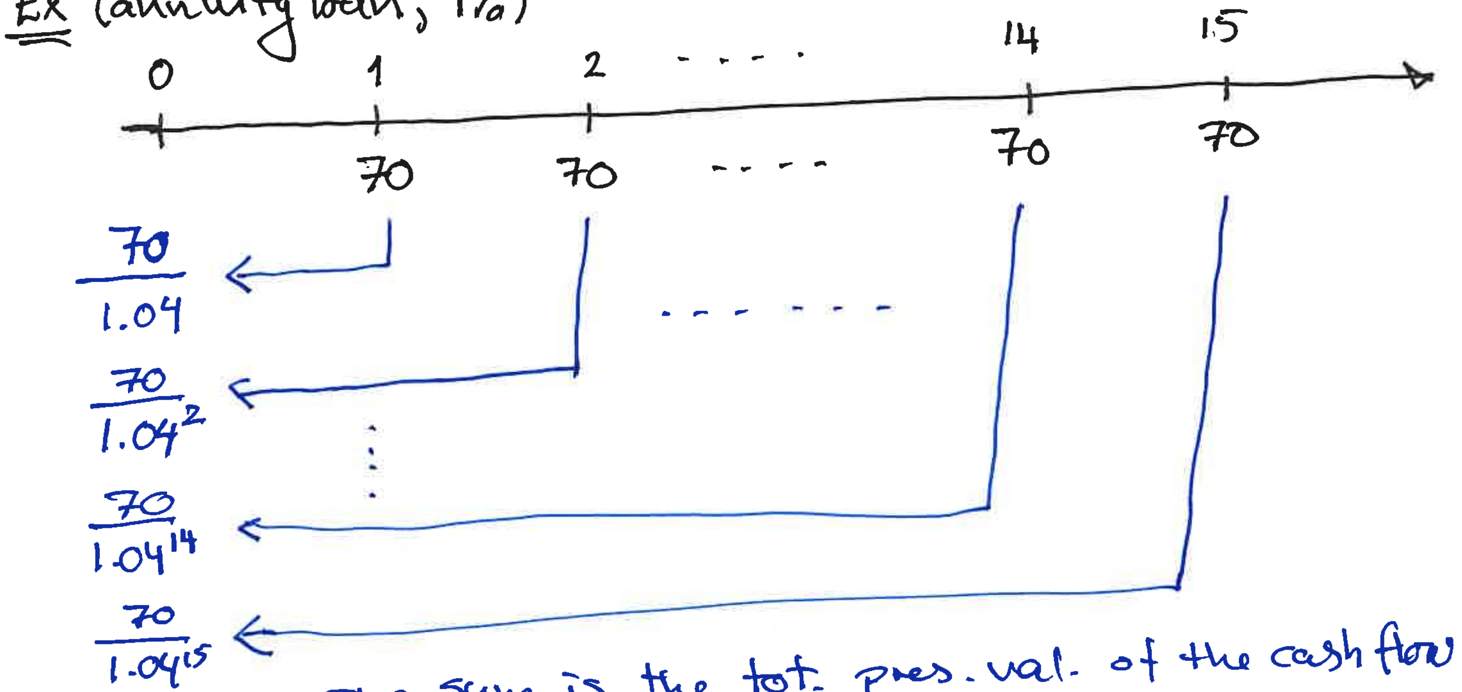
A fixed amount is paid each period.

Ex Annuity loan (tot. present value = what you can borrow)

Ex Saving with a fixed amount each period.
Future value: The balance (what you have saved)

The tot. present value and the future value of regular cash flows are geometric series.

Ex (annuity loan, 4%)



The sum is the tot. pres. val. of the cash flow = what you can borrow.

We get a geometric series :

$$\frac{70}{1.04} + \frac{70}{1.04^2} + \cdots + \frac{70}{1.04^{14}} + \frac{70}{1.04^{15}}$$

We read the geometric series backwards:

$$a_1 = \frac{70}{1.04^{15}}, \quad k = 1.04, \quad n = 15$$

Then the sum (the tot. pres. val.) is

$$a_1 \cdot \frac{k^n - 1}{k-1} = \frac{70}{1.04^{15}} \cdot \frac{1.04^{15} - 1}{0.04} = \underline{\underline{778.29}}$$

We could also read the sum forwards.

Then $$a_1 = \frac{70}{1.04}, \quad k = \frac{1}{1.04}, \quad n = 15$$

the sum is $$\frac{70}{1.04} \cdot \frac{\left(\frac{1}{1.04}\right)^{15} - 1}{\frac{1}{1.04} - 1} = \frac{70}{1.04} \cdot \frac{\left(1 - \frac{1}{1.04^{15}}\right)}{\left(1 - \frac{1}{1.04}\right)}$$

$$= \frac{70\left(1 - \frac{1}{1.04^{15}}\right)}{1.04 - 1} = \frac{70\left(1 - \frac{1}{1.04^{15}}\right)}{0.04}$$

$$= \underline{\underline{778.29}}$$

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2. Infinite series and limit values

Ex The annuity : 70 000

interest : 4 %

Number of years : n

First payment : 1 year from now.

The total present value :

$$\frac{70'}{1.04^n} \cdot \frac{1.04^n - 1}{0.04} = \frac{70' \cdot (1.04^n - 1)}{1.04^n \cdot 0.04}$$

$$= \frac{70' \cdot (1.04^n - 1) : 1.04^n}{\cancel{1.04^n} \cdot 0.04 : \cancel{1.04^n}}$$

$$= \frac{70' \cdot \left(\frac{1.04^n}{1.04^n} - \frac{1}{1.04^n}\right)}{0.04} = \frac{70'\left(1 - \frac{1}{1.04^n}\right)}{0.04}$$

approaches 0 when $n \to \infty$

What happens if
n approaches infinity
"becomes larger and larger"

$n \to \infty$

Then $$\frac{70'\left(1 - \frac{1}{1.04^n}\right)}{0.04} \xrightarrow[n \to \infty]{} \frac{70'(1-0)}{0.04}$$

$$= \frac{70'}{0.04} = \underline{\underline{1750'}}$$

<u>Conclusion</u> : If you pay the bank 70 000 each year forever, starting next year, with 4% interest, you can borrow 1.75 mill .

3. Euler's number and continuous compounding

Ex You deposit 1000 into an account with 12% nominal interest.

compounding	Balance after one year
Annual	$1000 \cdot 1.12 = 1120.00$
Half year	$1000 \cdot 1.06^2 = 1123.60$
Quarterly	$1000 \cdot 1.03^4 = 1125.51$
Monthly	$1000 \cdot 1.01^{12} = 1126.83$
Daily	$1000 \cdot \left(1+\frac{0.12}{365}\right)^{365} = 1127.47$
Pattern (n periods)	$1000 \cdot \left(1+\frac{0.12}{n}\right)^n$

!!

Euler's number: $e = 2.71828\ldots$

1 $\boxed{e^x}$

Calculate: $1000 \cdot e^{0.12} = 1127.50$

1000 $\boxed{\times}$ 0.12 $\boxed{e^x}$ $\boxed{=}$

Euler's number is defined as the limit of $\left(1+\frac{1}{n}\right)^n$ when n approaches ∞ ('becomes bigger and bigger')

Write: $\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$

Ex

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.71692\ldots$$

$$\left(1+\frac{1}{1\,\text{mill}}\right)^{1\,\text{mill}} = 2.718280\ldots$$

Back to the ex. with $r = 12\%$.

$$\left(1+\frac{0.12}{n}\right)^n = \left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^n$$

$$= \Bigg[\underbrace{\left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{\frac{n}{0.12}}}_{\substack{\text{approaches } e \\ \text{when } n \to \infty}}\Bigg]^{0.12}$$

so $\left(1+\frac{0.12}{n}\right)^n \xrightarrow[n\to\infty]{} e^{0.12}$

and $1000 \cdot \left(1+\frac{0.12}{n}\right)^n \longrightarrow 1000 \cdot e^{0.12}$

After 1 year with 12% nominal interest and <u>continuous compounding</u> the deposit of 1000 has increased to

$$1000 \cdot e^{0.12} = 1127.50$$

the annual growth factor!

The annual growth factor (with continuous compounding) is

$$e^{0.12} = 1.127497$$

The <u>effective</u> interest is

$$e^{0.12} - 1 = 12.7497\%$$

After 2 years :

$$1000 \cdot e^{0.12} \cdot e^{0.12} = 1000 \cdot \left(e^{0.12}\right)^2$$

$$= 1000 \cdot e^{0.12 \cdot 2} = 1000 \cdot e^{0.24}$$

$$= \underline{\underline{1271.25}}$$

<u>Problem</u> You deposit 10 mill into an account with nominal interest : 2.8%

Calculate the balance after 5 years with

a) Annual compounding

b) Continuous compounding

c) Compute the effective (annual) interest with continuous compounding.

Solution:

a) Annual growth factor: 1.028

Balance after 5 years: $10 \text{ mill} \cdot 1.028^5$

$= \underline{\underline{11.48 \text{ mill}}}$

b) Annual growth factor: $e^{0.028} = 1.0284$

Balance after 5 years: $10 \text{ mill} \cdot (e^{0.028})^5$

$= 10 \text{ mill} \cdot e^{0.028 \cdot 5}$

$= 10 \text{ mill} \cdot e^{0.140}$

$= \underline{\underline{11.50 \text{ mill}}}$.

c) The effective interest is $e^{0.028} - 1$

$= 1.0284 - 1 = \underline{\underline{2.84\,\%}}$.