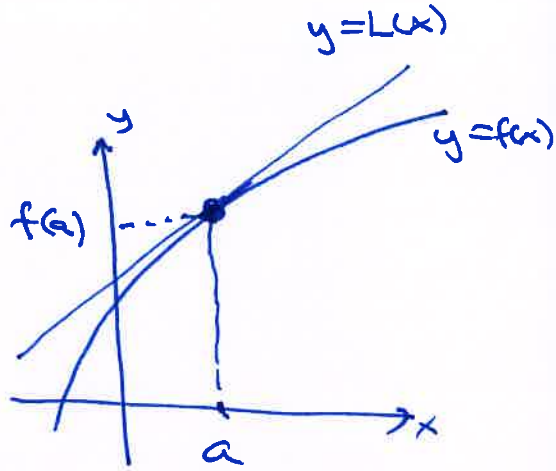
① Linear approximation

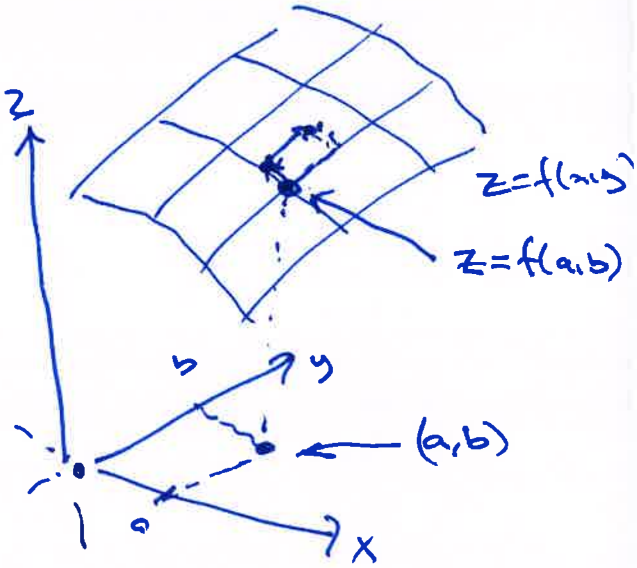


One variable: $y=f(x)$, $x=a$

$$L(x) = f(a) + f'(a)\cdot(x-a)$$

$$\boxed{\begin{aligned} y-y_0 &= k\cdot(x-x_0) \\ y &= y_0 + k\cdot(x-x_0)\end{aligned}}$$

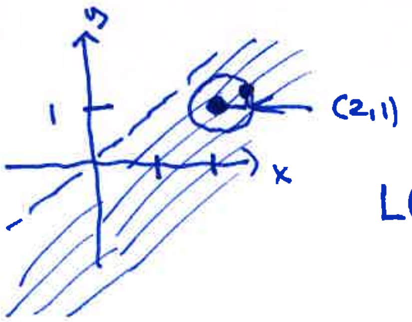
Two variables: $z=f(x,y)$, $(x,y)=(a,b)$



$$\boxed{\begin{aligned} L(x,y) = f(a,b) &+ f'_x(a,b)\cdot(x-a) \\ &+ f'_y(a,b)\cdot(y-b)\end{aligned}}$$

Ex: $f(x,y) = \ln(x-y)$, $x>y$

$(a,b) = (2,1)$



$$f'_x = \frac{1}{x-y}\cdot 1 = \frac{1}{x-y}$$

$$f'_y = \frac{1}{u}\cdot(-1) = \frac{-1}{x-y}$$

$$L(x,y) = f(2,1) + f'_x(2,1)\cdot(x-2) + f'_y(2,1)\cdot(y-b)$$

$$= 0 + \frac{1}{1}\cdot(x-2) + \frac{-1}{1}\cdot(y-1)$$

$$= (x-2) - (y-1) = \underline{\underline{x-y-1}}$$

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② Problems:

Exam 12/2017 MET1180: Q 4

$f(x,y) = x^2y^2 + xy + x - y$

a) $f'_x = 2xy^2 + y + 1$

$f'_y = 2x^2y + x - 1$

$f''_{xx} = 2y^2 \quad f''_{xy} = 4xy+1$

$f''_{yx} = 4xy+1, \quad f''_{yy} = 2x^2$

$$H(f) = \begin{pmatrix} 2y^2 & 4xy+1 \\ 4xy+1 & 2x^2 \end{pmatrix}$$

b) $L_1: f(x,y) = 2$

$L_2: x = y$

$$\boxed{\begin{array}{r} x^2y^2 + xy + x - y = 2 \\ x = y \end{array}}$$

$y = x \Rightarrow y^4 + y^2 + 0 = 2$

$y^4 + y^2 - 2 = 0$

$u = y^2$: $u^2 + u - 2 = 0$

$(u+2)(u-1) = 0$

$u = -2$ or $u = 1$

$\cancel{y^2 = -2}$ or $y^2 = 1$

$y = \pm 1$

Intersection points:

$(x,y) = (1,1), (-1,-1)$

c) Tangents:

$(1,1)$: $y - 1 = k_1(x-1)$

$y - 1 = -2(x-1)$

$y = -2x + 3$

$(-1,-1)$: $y + 1 = k_2(x+1)$

$y + 1 = -\frac{1}{2}(x+1)$

$y = -\frac{1}{2}x - \frac{1}{2} - 1$

$y = -\frac{1}{2}x - \frac{3}{2}$

$f(x,y) = 2$

$x^2y^2 + xy + x - y = 2$

$f'_x + f'_y \cdot y' = 0$

$$y' = -\frac{f'_x}{f'_y} = -\frac{2xy^2 + y + 1}{2x^2y + x - 1}$$

$k_1 = y'(1,1) = -\frac{4}{2} = -2$

$k_2 = y'(-1,-1) = -\frac{-2-1+1}{-2-1-1}$

$= -\frac{-2}{-4} = -1/2$

d) $f'_x = \boxed{2xy^2 + y + 1 = 0}$ FOC
$f'_y = \boxed{2x^2y + x - 1 = 0}$

Alt A

$$x = \frac{2xy^2}{2y^2} = -y-1 = \frac{-(y+1)}{2y^2}$$

$$\frac{\cancel{2}(y+1)^2}{\cancel{4}_2\, y^{\cancel{4}_3}}\cancel{y} - \frac{y+1}{2y^2} - 1 = 0 \quad | \cdot 2y^3$$

$$(y+1)^2 - (y+1)\cdot y - 2y^3 = 0$$

$$\cancel{y^2} + 2y + 1 - \cancel{y^2} - y - 2y^3 = 0$$

$$-2y^3 + y + 1 = 0 \quad | (-1)$$

$$\boxed{2y^3 - y - 1 = 0}$$

$y = 1$ is sol.

$$(2y^3 - y - 1) : (y-1) = 2y^2 + 2y + 1$$
$$\underline{2y^3 - 2y^2}$$
$$2y^2 - y - 1$$
$$\underline{2y^2 - 2y}$$
$$y - 1$$
$$\underline{y - 1}$$
$$0$$

$\Rightarrow (y-1)\cdot(2y^2 + 2y + 1) = 0$

$\underline{y = 1}$ or $2y^2 + 2y + 1 = 0$

$$y = \frac{-2 \pm \sqrt{4 - 4\cdot 2\cdot 1}}{2\cdot 2}$$

no solutions

Alt B

Add the equations:

$$2xy^2 + 2x^2y + y + x + \cancel{1} - \cancel{1} = 0$$

$$2xy(y+x) + (y+x) = 0$$

$$(y+x)(2xy+1) = 0$$

$\underline{x = -y}$ or $\underline{2xy = -1}$

$2y^3 - y - 1 = 0$
$\Updownarrow$
$y = 1, x = -1$
$\Downarrow$
$(x,y) = \underline{(-1,1)}$

$2xy^2 + y + 1 = 0$
$\underline{2xy}\cdot y + y + 1 = 0$
$\cancel{-y} + \cancel{y} + 1 = 0$
$1 = 0$
no solutions

Stationary pts:

$\underline{y = 1}$ $\underline{x = -1}$ $\Rightarrow (x,y) = \underline{\underline{(-1,1)}}$

$$H(f)(-1,1) = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

$\det = AC - B^2 = 4 - 9 = -5 < 0$

$(-1,1)$ saddle pt

Exam MET1180, 05/2017 Q4

$D_f: xy \neq 0$

$$f(x,y) = \frac{2x+3y-6}{xy} = \frac{2\cancel{x}}{\cancel{x}y} + \frac{3\cancel{y}}{x\cancel{y}} - \frac{6}{xy} = 2y^{-1} + 3x^{-1} - 6x^{-1}y^{-1}$$

a) $$f'_x = -3x^{-2} - 6y^{-1}\cdot(-1)x^{-2} = \frac{-3\cdot y}{x^2\cdot y} + \frac{6}{x^2 y} = \frac{6-3y}{x^2 y}$$

$$f'_y = -2y^{-2} - 6x^{-1}(-1)y^{-2} = \frac{-2\cdot x}{y^2\cdot x} + \frac{6}{xy^2} = \frac{6-2x}{xy^2}$$

$f'_x = f'_y = 0$:

$$\frac{6-3y}{x^2y} = 0 \Rightarrow 6-3y=0 \Rightarrow y=2$$

$$\frac{6-2x}{xy^2} = 0 \Rightarrow 6-2x=0 \Rightarrow x=3$$

Stat. pts: $(x,y) = (3,2)$

b) $$f''_{xx} = 6x^{-3} + 6y^{-1}\cdot(-2)x^{-3} = \frac{6\cdot y}{x^3\cdot y} - \frac{12}{x^3y} = \frac{6y-12}{x^3y}$$

$$f''_{xy} = 6x^{-2}(-1)y^{-2} = \frac{-6}{x^2y^2}$$

$$f''_{yy} = 4y^{-3} + 6x^{-1}(-2)y^{-3} = \frac{4\cdot x}{y^3\cdot x} - \frac{12}{xy^3} = \frac{4x-12}{xy^3}$$

$$H(f) = \begin{pmatrix} \frac{6y-12}{x^3y} & \frac{-6}{x^2y^2} \\ \frac{-6}{x^2y^2} & \frac{4x-12}{xy^3} \end{pmatrix} \Rightarrow H(f)(3,2) = \begin{pmatrix} 0 & -1/6 \\ -1/6 & 0 \end{pmatrix}$$

$$\det = 0^2 - (-1/6)^2 = -\frac{1}{36} < 0$$

$(x,y) = (3,2)$ is saddle pt

c) f has no global max/min

one stationary pt, that is saddle pt $\Rightarrow$ not max/min

$D_f: xy \neq 0 \Rightarrow$ no boundary pts in D_f,

f'_x, f'_y are defined for all pts in D_f

d) $\underline{f(x,y) = 5}$:

$$\frac{2x+3y-6}{xy} = 5$$

$$y = 1$$

$$\Downarrow$$

$$\frac{2x+3\cdot 1-6}{x\cdot 1} = 5 \quad | \cdot x$$

$$2x - 3 = 5x$$

$$-3x = 3$$

$$\underline{x = -1}$$

$$\boxed{\begin{array}{l}\underline{\text{Intersection pts:}} \\ (x,y) = \underline{\underline{(-1,1)}}\end{array}}$$

$\underline{\text{Tangent}}$: $y - 1 = k \cdot (x+1)$

$$y' = -\frac{f'_x}{f'_y} = -\frac{\frac{6-3y}{x^2y} \cdot x^2y^2}{\frac{6-2x}{xy^2} \cdot x^2y^2} = -\frac{y(6-3y)}{x(6-2x)}$$

$$\underline{y'(-1,1) = -\frac{1\cdot 3}{(-1)\cdot 8} = \underline{3/8}}$$

$\Rightarrow$ $y - 1 = 3/8\,(x+1)$

$$y = \frac{3}{8}x + \frac{3}{8} + 1$$

$$\underline{\underline{y = \frac{3}{8}x + \frac{11}{8}}}$$

Problem set 22

5.

a) $f(x,y) = xy(x^2-y^2) = x^3y - xy^3$

$f'_x = \underline{3x^2y - y^3} = 0$ $\qquad y(3x^2-y^2)=0$

$f'_y = \underline{x^3 - 3xy^2} = 0$ $\qquad x(x^2-3y^2)=0$

$y=0$ or $3x^2=y^2$

and

$x=0$ or $x^2=3y^2$

1) $y=0, x=0$: $(0,0)$

2) $y=0, x^2=3y^2$ "

3) $3x^2=y^2, x=0$: "

4) $3x^2=y^2, x^2=3y^2$: "

$y^2=3x^2 \Rightarrow x^2=3\cdot(3x^2)$

$x^2=9x^2$

Stat. pts: $(x,y)=(0,0)$

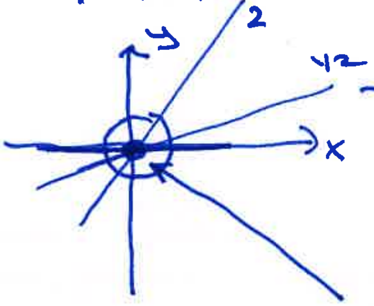
$$H(f)=\begin{pmatrix} 6xy & 3x^2-3y^2 \\ 3x^2-3y^2 & -6xy \end{pmatrix}$$

$$H(f)(0,0)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

det $= 0 \Rightarrow$ second derivative test inconclusive

Use defn:

$f(0,0)=0$



$f(0,0)=0$

$f(x,y)=xy(x^2-y^2)$

$(0,0)$ local min : $f(x,y)\geq 0$ for all (x,y) close to $(0,0)$

local max : $f(x,y)\leq 0$ — " —

saddle pt : all other cases

$\underline{x=2y}$:

$(x=2a, y=a)$: $f(2a,a) = 2a\cdot a(4a^2-a^2) = 2a^2\cdot 3a^2 = 6a^4 \geq 0$

$\underline{y=2x}$:

$(x=a, y=2a)$: $f(a,2a) = a\cdot 2a\cdot(a^2-4a^2) = 2a^2(-3a^2) = -6a^4 \leq 0$

$\Downarrow$

$(0,0)$ is a saddle pt

$y=0$: x-axis

$(x,y)=(a,0)$: $f(a,0)=0$

b) See Lecture 21, Part 2 for computation of stationary pts

c) $f(x,y) = \sqrt{u}$, $u = 36 - 9x^2 - 4y^2$

$f'_x = \frac{1}{2\sqrt{u}} \cdot u'_x = \frac{-18x}{2\sqrt{u}} = 0$ $\quad -18x = 0 \Rightarrow x = 0$

$f'_y = \frac{1}{2\sqrt{u}} \cdot u'_y = \frac{-8y}{2\sqrt{u}} = 0$ $\quad -8y = 0 \Rightarrow y = 0$

$\Rightarrow$ Stat. pt: $(x,y) = \underline{\underline{(0,0)}}$

Second derivative test.

$f(0,0) = \underline{6}$

$f(x,y) = \sqrt{36 - 9x^2 - 4y^2} \leq \sqrt{36} = 6$ for all x,y

$\Downarrow$

$(0,0)$ is global max for f

$\Downarrow$

$(0,0)$ is local max for f