Plan

1 Exam problems
2 Revision

Reminder:
Do the course evaluation.

① Revision key topics

(a) Integration:

basics + techniques (defn. power rule)

$\begin{cases} \text{– substitution} \\ \text{– integration by parts} \\ \text{– partial fractions} \end{cases}$

(b) Hessian matrix: $f(x,y)$ fn. in two variables

$$H(f)(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix} \qquad H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$$

$f''_{yx}(x,y) = f''_{xy}(x,y)$

Symmetric matrix

Second derivative test:

(x^*,y^*) stat pt for f $\longrightarrow$ $H(f)(x^*,y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ $\quad \det = AC - B^2$, $\mathrm{tr} = A + C$

$$\boxed{\begin{array}{lll} \det > 0,\ \mathrm{tr} > 0: & (x^*,y^*) & \text{local min} \\ \det > 0,\ \mathrm{tr} < 0: & \text{"} & \text{local max} \\ \det < 0 : & \text{"} & \text{saddle pt.} \end{array}}$$

(c) Limits: $a_0 + a_0 \cdot k + a_0 \cdot k^2 + \dots = a_0 \cdot \frac{1}{1-k}$ if $-1<k<1$

$= \lim_{n \to \infty} S_n$

$$\left(S_n = a_0 + a_0 k + a_0 k^2 + \dots + a_0 k^{n-1} = a_0 \cdot \frac{1-k^n}{1-k} \quad (n = \#\text{terms}) \right)$$

How to compute limits: Ex $\lim_{x\to-\infty} x e^{x}$ ($x \to -\infty$, $e^x = \frac{1}{e^{-x}} \to 0$)

L'Hôpitals rule for "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$0\cdot\infty$"

$$\lim_{x\to-\infty} x e^{x} = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \underline{\underline{0}}$$

Compounding n times a year: interest rate r / year

$\underline{n=1}$: $1+r$ ← growth factor per year $\qquad B_1 = B_0 \cdot (1+r)$

$\underline{n=12}$: $\left(1+\frac{r}{12}\right)^{12}$ — " — $\qquad B_1 = B_0 \cdot \left(1+\frac{r}{12}\right)^{12}$

$\underline{n\to\infty}$: $\underbrace{\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}}_{e^r}$ growth factor per year w/ cont. comp.

$$e = \text{Euler's number} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 2.71828\ldots$$

(d) Some important plane curves

i) Ellipse, center (x_0, y_0), half-axis $a, b > 0$: $\qquad \dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$

ii) Circle, center (x_0, y_0), radius $r > 0$ $\qquad (x-x_0)^2 + (y-y_0)^2 = r^2$

$x^2+4y^2=36$ ellipse $\qquad 2x^2+2y^2=36$ circle $\qquad xy=4$ hyperbola

iii) Hyperbola:

$x=c$: vertical asymptote

$y=a$: horizontal asymptote

$$y-a = \frac{b}{x-c}$$
$$\Updownarrow$$
$$(x-c)(y-a) = b$$
$$\Updownarrow$$
$$\underbrace{y = a + \frac{b}{x-c}}\; = \frac{a(x-c)+b}{x-c}$$

iv) Parabola: $y - y_0 = a \cdot (x - x_0)^2$

v) Straight lines: $y - y_0 = a \cdot (x - x_0)$

e) Finding asymptotes

$f(x) = \dfrac{p(x)}{q(x)}$

rational function

i) Vertical asymptotes:

$x = c \iff q(c) = 0,\ p(c) \neq 0$

ii) Horisontal/slant asymptotes:

$\dfrac{p(x)}{q(x)} = \underbrace{a(x)}_{\text{polynomial division}} + \dfrac{b(x)}{q(x)}$

$x \to \pm\infty \implies \dfrac{b(x)}{q(x)} \to 0$

Asymptote:
$y = a(x)$

f) Functions in one variable

max/min-problems: max/min $f(x)$

* Stat. pts: $f'(x) = 0$
* pts where $f'(x)$ does not exist
* boundary pts

} cand. for max/min

* Sign diagram for $f'(x)$
* convex/concave fn.

9) Stationary pts:

$f(x)$ one var: $f'(x)=0$

$f(x,y)$ two var: $\begin{cases} f'_x(x,y)=0 \\ f'_y(x,y)=0 \end{cases}$

candidates for max/min

10) Gaussian elimination

Ex:

$$\begin{aligned} x+3y-5z&=7 \\ 2x-y+z&=3 \\ 5x+y-z&=10 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 3 & -5 & 7 \\ 2 & -1 & 1 & 3 \\ 5 & 1 & -1 & 10 \end{array}\right)$$

linear system

extended matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & -5 & 7 \\ 0 & -7 & 11 & -11 \\ 0 & 0 & 2 & -3 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 3 & -5 & 7 \\ 0 & -7 & 11 & -11 \\ 0 & -14 & 24 & -25 \end{array}\right)$$

echelon form

$$\begin{aligned} x+3y-5z&=7 \\ -7y+11z&=-11 \\ 2z&=-3 \end{aligned}$$

$z=-3/2$

$-7y=-11-11(-3/2) = \dfrac{-22+33}{2} = \dfrac{11}{2}$

$y=-\dfrac{11}{14}$

$x = 7-3\left(-\frac{11}{14}\right)+5\left(-\frac{3}{2}\right) = \dfrac{7\cdot 14+33-15\cdot 7}{14} = \dfrac{26}{14} = \dfrac{13}{7}$

One solution:

$(x,y,z) = (13/7, -11/14, -3/2)$

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5. $f(x,y) = y^2 - x^3 + 3x$

C: level curve of f going through $(-1,2)$

a) Stationary pts

$f'_x = -3x^2+3 = 0 \Rightarrow 3x^2 = 3$, $x^2 = 1 \Rightarrow x = \pm 1$

$f'_y = 2y = 0 \Rightarrow y = 0$

Stat pts: $(1,0), (-1,0)$

$$H(f) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix}$$

$$H(f)(1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \quad \det = -12 < 0$$

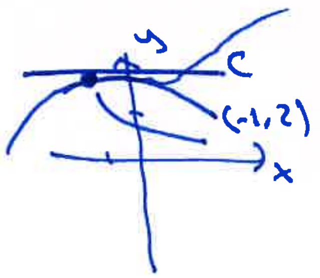
$(1,0)$ is saddle pt

$$H(f)(-1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \quad \det = 12 > 0, \quad \operatorname{tr} = 8 > 0$$

$(-1,0)$ loc min

b) C: level curve of f thr. $(-1,2)$

$f(-1,2) = 2 \Rightarrow$ C: $f(x,y) = 2$



Tangent line: $y - y_0 = a\cdot(x - x_0)$

$y - 2 - 0\,(x+1) = 0 \Rightarrow y = 2$

$$y' = -\frac{f'_x}{f'_y} = -\frac{-3x^2+3}{2y} = \frac{3x^2-3}{2y}$$

$$a = y'(-1,2) = \frac{3\cdot(-1)^2 - 3}{2\cdot 2} = \frac{0}{4} = 0$$

Intersection: $y = 2$, $f(x,y) = 2$; $y = 2$, $y^2 - x^3 + 3x = 2$

$2^2 - x^3 + 3x = 2$

$x^3 - 3x = 2$

$x^3 - 3x - 2 = 0$

$x^3-3x-2=0 \quad \Leftarrow \quad x=-1$ is a solution

$x^3-3x-2 : (x+1) = x^2-x-2$

x^3+x^2

$-x^2-3x-2$

$-x^2-x$

$-2x-2$

$-2x-2$

0

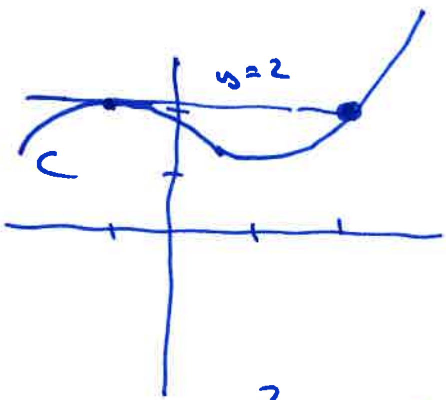
$x^3-3x-2=0$

$(x+1)(x^2-x-2)=0$

$\underline{x=-1}$ or $x=\dfrac{1\pm\sqrt{1+8}}{2} = \dfrac{1\pm 3}{2} = \underline{2}, \underline{-1}$

Other intersection pts:

$\underline{\underline{x=2}}$ $\quad \underline{\underline{(2,2)}}$



c) $4x^2+y^2=4 \quad | :4 \quad \Rightarrow \quad x^2+\dfrac{y^2}{4}=1$

ellipse

$\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}=1$

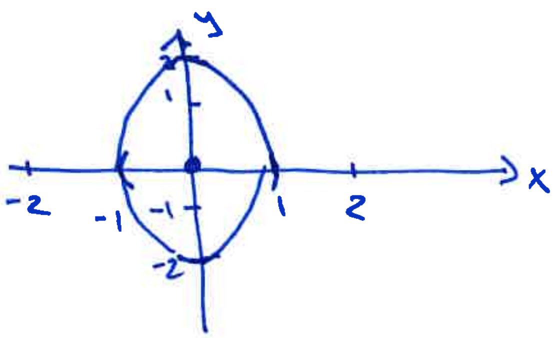
$\dfrac{x^2}{1^2}+\dfrac{y^2}{2^2}=1$

ellipse
center $(0,0)$
half-axes
$a=1, b=2$

Yes, bounded since

$-1 \le x \le 1$

$-2 \le y \le 2$



d) Solve: max $f(x,y) = y^2 - x^3 - 3x$ when $4x^2+y^2=4$

i) Find candidate pts using Lagrange multipliers

$L = y^2 - x^3 + 3x - \lambda(4x^2+y^2-4)$

$$\boxed{\begin{aligned} L'_x &= -3x^2+3-\lambda\cdot(8x) = 0 \\ L'_y &= 2y - \lambda\cdot(2y) = 0 \\ & 4x^2+y^2 = 4 \end{aligned}}$$

Lagrange conditions.

(1) $-3x^2+3 = \lambda\cdot(8x)$

$\lambda = \dfrac{-3x^2+3}{8x}$

(2) $2y = \lambda\cdot 2y$

$\lambda = \dfrac{2y}{2y} = 1$

$\Downarrow$

$1 = \dfrac{-3x^2+3}{8x} \quad |\cdot 8x$

$8x = -3x^2+3$

$3x^2+8x-3=0$

$x = \dfrac{-8\pm\sqrt{64-4\cdot3\cdot(-3)}}{2\cdot3}$

$= \dfrac{-8\pm10}{6} = \dfrac{2}{6}, -3$

$x = 1/3$ or $x=-3$

check: $8x=0$

$x=0$

(1) $3-\lambda\cdot0=0$ $\lambda=0$

(2) $2y=0$ $y=0$

(3) $0=4$ impossible

check: $2y=0$ $y=0$

(3) $4x^2=4$ $x=\pm1$

(2) $0=0$

(1) $-\lambda\cdot(8\cdot(\pm1))=0$

$\lambda=0$

$\Downarrow$

Pts: $(\pm1, 0; 0)$

$\underline{x = 1/3}$:
$$\lambda = \frac{-3x^2+3}{8x} = \frac{-3\cdot(1/3)^2+3}{8\cdot 1/3}\ \frac{\cdot 3}{\cdot 3} = \frac{-1+9}{8} = 1$$

$$4x^2+y^2=4 \qquad 4(1/3)^2+y^2=4$$
$$y^2 = 4 - 4/9 = 32/9$$
$$y = \pm\sqrt{32}/3$$

$\Downarrow$

Pts: $\underline{(1/3, \pm\sqrt{32}/3 ; 1)}$

$\underline{x=-3}$: $4x^2+y^2=4$
$$4\cdot(-3)^2+y^2=4$$
$$y^2 = 4-36 = -32$$
impossible

$\Downarrow$

Candidate pts: $(x,y;\lambda) = (\pm 1, 0; 0)$,
$(1/3, \pm\sqrt{32}/3 ; 1)$

ii) Solve the Lagrange pb: max y^2-x^3+3x when $4x^2+y^2=4$

(a) $4x^2+y^2=4$ (ellipse) $\xRightarrow{\text{EVT}}$ there is a maximum
(from c)

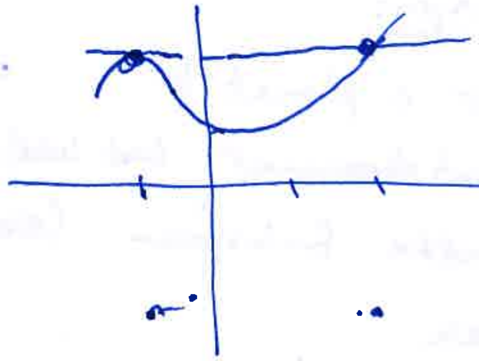
(b) no adm pts with degenerate constraints
$(g'_x = g'_y = 0 \quad g = 4x^2+y^2)$

(c) $f(1,0)=2$ $\quad f(1/3, \pm\sqrt{32}/3) = 32/9 - 1/27 + 1$
$f(-1,0)=-2$ $\quad = \dfrac{32\cdot 3 - 1 + 27}{27} = \dfrac{122}{27}$

$\Downarrow$

$f_{max} = \underline{\underline{\frac{122}{27}}}$ at $\underline{\underline{(1/3, \pm\sqrt{32}/3)}}$ with $\lambda = 1$

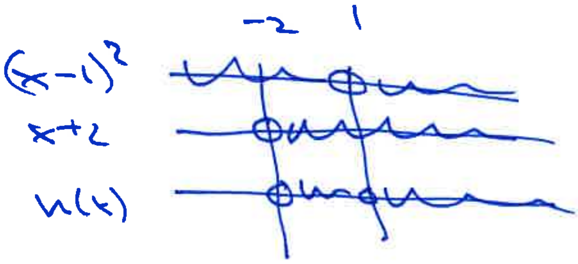
6. min $f(x,y) = x$ when $\underbrace{y^2 - x^3 + 3x = 2}_{\text{curve } C}$

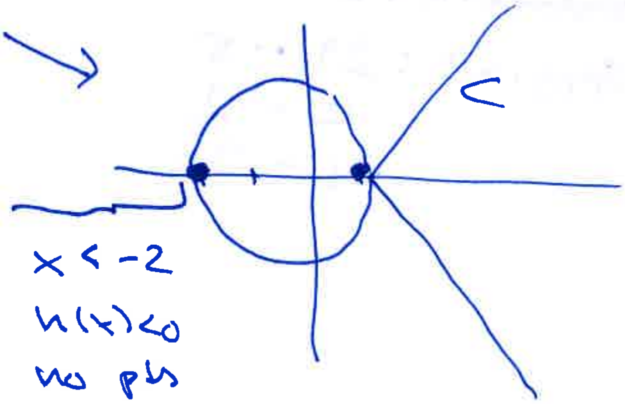


When is $h(x) < 0$?

$h(x) = (x^3 - 3x + 2)$
$= (x-1)(x^2 + x - 2)$
$= (x-1)(x-1)(x+2)$
$= (x-1)^2(x+2)$

$y^2 = \underbrace{x^3 - 3x + 2}_{h(x)}$





$x < -2$
$h(x) < 0$
no pts

$f_{min} = -2$

at $(-2, 0)$