EBA2911 Mathematics for Business Analytics autumn 2020

Exercises

... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.

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Lecture 10

on Wednesday 7 Sept. 10-11.45 in B2-060

Sec. 5.3, 4.9-10: Inverse functions. Exponential functions. Logarithms.

Here are recommended exercises from the textbook [SHSC].

Section **5.3** exercise 1, 3-5, 7, 9, 10
Section **4.9** exercise 1-4, 8-10
Section **4.10** exercise 1, 2, 6

Problems for the exercise session Wednesday 7 Oct. at 12-15 in CU1-067 or on Zoom

Problem 1 Suppose $g(x)$ is the inverse function of $f(x)$. Determine:

a) $g(10)$ if $f(3) = 10$ b) $f(g(5))$ c) $f(\sqrt{2})$ if $g(3) = \sqrt{2}$

d) $g(f(9))$

Problem 2 Determine the inverse function $g(x)$ and the domain D_g of the function $f(x)$ with domain D_f.

a) $f(x) = 2x - 3$ with D_f = all numbers

b) $f(x) = 0.5x + 1.5$ with D_f = all numbers

c) $f(x) = x^2 + 6x$ with $D_f = \langle \leftarrow, -3]$

d) $f(x) = 20 + \frac{1}{x-3}$ with $D_f = \langle 3, \rightarrow \rangle$

e) $f(x) = (x-1)^3 + 50$ with $D_f = [1, \rightarrow \rangle$

Problem 3 We have (approximately) $\ln 2 = 0.6931$ and $\ln 3 = 1.0986$ and $\ln 5 = 1.6094$. Use these numbers to determine the values (approximately) without using the ln-button on the calculator.

a) $\ln 250$ b) $\ln 625$ c) $\ln \frac{625}{216}$

d) $\ln \frac{1\,000\,000}{27}$ e) $\ln 130 - \ln 78$ f) $\ln \sqrt[10]{6}$

Problem 4 Solve the equations.

a) $e^x = 5$ b) $e^{2x+1} = 5$ c) $e^{2x+1} = 3e^{x+2}$

d) $\ln(x) = -2$ e) $\ln(7x - 3) = -2$ f) $\ln(x - 3) = \ln(2x + 1) + 1$

g) $e^{2x} - 4e^x - 5 = 0$

Problem 5 Solve the inequalities.

a) $e^x \geqslant 5$ b) $e^{2x+1} \geqslant 5$ c) $\ln(x) < -2$ d) $\ln(x - 3) < -2$

e) $\frac{3e^x}{e^x+1} < 5$ f) $\ln \frac{3x-2}{x-7} \geqslant 0$

Problem 6 Determine the asymptotes of the function.

a) $f(x) = e^{-0.1x} + 23$ b) $f(x) = e^{x(10-x)} + 50$ c) $f(x) = \frac{100e^{0.04x}}{e^{0.04x}+50}$

d) $f(x) = \ln(10-x)$ e) $f(x) = \ln(x^2 - 400)$

f) $f(x) = \ln(120x + 10) - \ln(20x - 30)$, $D_f = \langle \frac{3}{2}, \rightarrow \rangle$

Problem 7 Determine the inverse function $g(x)$ and the domain D_g of the function $f(x)$ with domain D_f.
(a) $f(x) = e^{\frac{x}{3}} - 1$ with $D_f = [0, \rightarrow\rangle$ (b) $f(x) = 4\ln(x-10)$ with $D_f = [11, \rightarrow\rangle$

Problem 8 Determine the expression $f(x) = c + \frac{a}{x-b}$ of the hyperbolas (a-d) in figure 1.

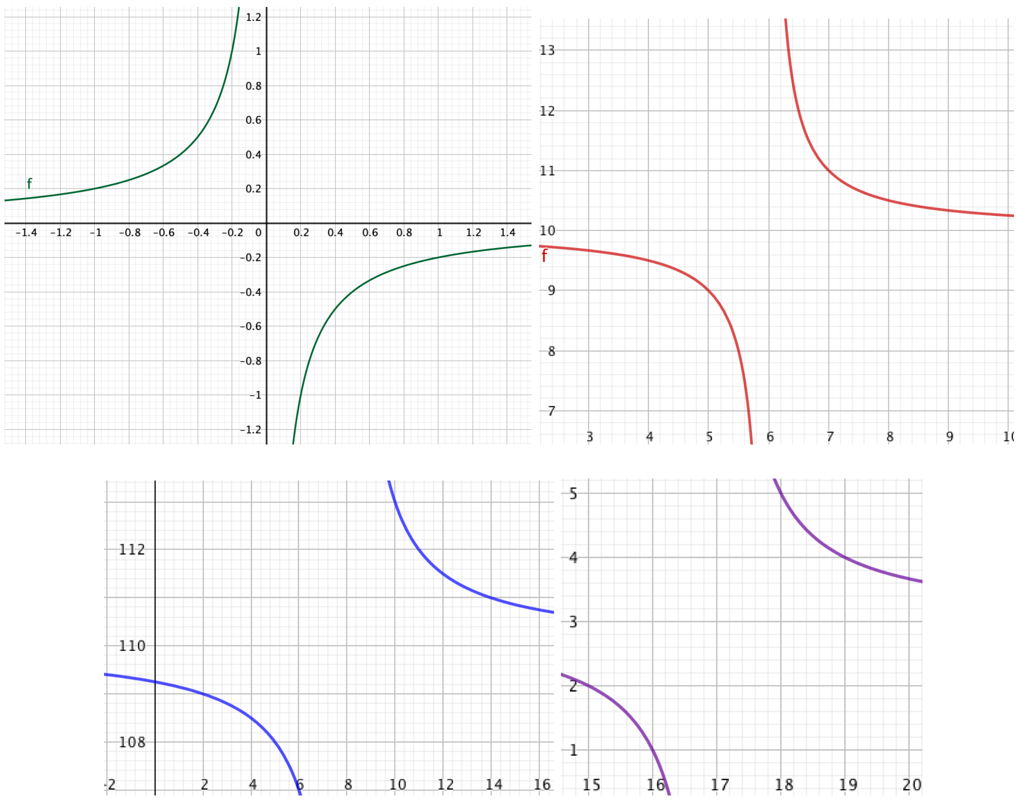

Figure 1: Hyperbolas a-d

Problem 9 Determine the asymptotes of the hyperbolas (a-d) in Problem 8.
Problem 10 Determine the asymptotes of the rational functions.

a) $f(x) = \frac{4x-10}{x-3}$ b) $f(x) = \frac{70-40x}{3-2x}$ c) $f(x) = \frac{12}{x^2+3}$

d) $f(x) = \frac{4x^2-28x+40}{x^2-4x+3}$ e) $f(x) = \frac{x^2+3x+5}{x-7}$ f) $f(x) = \frac{x^3-8}{x^2-10x+16}$

Problem 11 Determine if the function $f(x)$ has a zero in the interval I. Hint: The intermediate value theorem!
a) $f(x) = \sqrt{x-2} - x + 3$ and $I = [4, 5]$
b) $f(x) = (x-5)\sqrt{(0.2x+5)} - 0.2(x-3)^2$ and $I = [5, 15]$
c) $f(x) = \frac{4x-10}{x-3} - 4$ and $I = [2, 4]$

Answers

Problem 1

a) 3 b) 5 c) 3 d) 9

Problem 2

a) $g(x) = 0.5x + 1.5$ with D_g = all numbers

b) $g(x) = 2x - 3$, D_g = all numbers

c) $g(x) = -3 - \sqrt{x+9}$, $D_g = R_f = [-9, \rightarrow\rangle$

d) $g(x) = 3 + \frac{1}{x-20}$, $D_g = \langle 20, \rightarrow\rangle$

e) $g(x) = \sqrt[3]{x-50} + 1$, $D_g = [50, \rightarrow\rangle$

Problem 3

a) $\ln 250 = \ln 2 + 3\ln 5 = 0.6931 + 3 \cdot 1.6094 = 5.5213$

b) $\ln 625 = 4\ln 5 = 4 \cdot 1.6094 = 6.4376$

c) $\ln \frac{625}{216} = 4\ln 5 - 3(\ln 3 + \ln 2) = 4 \cdot 1.6094 - 3(1.0986 + 0.6931) = 1.0625$

d) $\ln \frac{1\,000\,000}{27} = 6(\ln 5 + \ln 2) - 3\ln 3 = 6 \cdot (1.6094 + 0.6931) - 3 \cdot 1.0986 = 10.5192$

e) $\ln 130 - \ln 78 = \ln 5 + \ln 26 - \ln 3 - \ln 26 = 1.6094 - 1.0986 = 0.5108$

f) $\ln 6^{\frac{1}{10}} = \frac{1}{10} \cdot \ln 6 = \frac{1.0986+0.6931}{10} = 0.1792$

Problem 4

a) $x = \ln 5$ b) $x = \frac{1}{2}(\ln(5) - 1)$ c) $x = 1 + \ln(3)$

d) $x = e^{-2}$ e) $x = \frac{e^{-2}+3}{7}$ f) $x = -\dfrac{e+3}{2e-1}$

g) $x = \ln 5$

Problem 5

a) Because $\ln x$ is a strictly increasing function for $x > 0$ we can insert the left hand side and the right hand side into $\ln x$ and keep the inequality. It gives $x \geqslant \ln 5$.
b) We insert the left hand side and the right hand side into $\ln x$ and keep the inequality. It gives $x \geqslant \frac{1}{2}(\ln 5 - 1)$.
c) Because e^x is a strictly increasing function we can insert the left hand side and the right hand side into e^x and keep the inequality. It gives $0 < x < e^{-2}$.
d) We insert the left hand side and the right hand side into e^x and keep the inequality. It gives $3 < x < 3 + e^{-2}$.
e) All numbers on the number line (are called the real numbers and written as $\mathbb{R}$, i.e. $x \in \mathbb{R}$).
f) Note that the inequality only is defined for $x < \frac{2}{3}$ and for $x > 7$. We insert the left and right hand side into e^x and keep the inequality. This gives $\frac{3x-2}{x-7} \geqslant 1$ which we then solve: $x \leqslant -\frac{5}{2}$ or $x > 7$ (and this is within the domain of definition of the inequality). Alternate way of writing: $x \in \langle \leftarrow, -\frac{5}{2}] \cup \langle 7, \rightarrow\rangle$.

Problem 6

a) horizontal asymptote: $y = 23$ (when $x \to \infty$)

b) horizontal asymptote: $y = 50$ (when $x \to \infty$)

c) horizontale asymptotes: $y = 100$ ($x \to \infty$) and $y = 0$ ($x \to -\infty$)

d) vertical asymptote: $x = 10$ ($y \to -\infty$ when $x \to 10^-$)

e) vertical asymptotes: $x = \pm 20$ ($y \to -\infty$ when $x \to -20^-$ and $y \to -\infty$ when $x \to 20^+$)

f) vertical asymptote: $x = \frac{3}{2}$, horizontal asymptote: $y = \ln 6$

Problem 7

a) $g(x) = 3\ln(x+1)$, $D_g = R_f = [0, \rightarrow\rangle$

b) $g(x) = e^{\frac{x}{4}} + 10$, $D_g = [0, \rightarrow\rangle$

Problem 8

a) $f(x) = -\frac{1}{5x}$ b) $f(x) = 10 + \frac{1}{x-6}$ c) $f(x) = 110 + \frac{6}{x-8}$ d) $f(x) = 3 + \frac{2}{x-17}$

Problem 9

a) vertical asymptote: $x = 0$, horizontal asymptote: $y = 0$
b) vertical asymptote: $x = 6$, horizontal asymptote: $y = 10$
c) vertical asymptote: $x = 8$, horizontal asymptote: $y = 110$
d) vertical asymptote: $x = 17$, horizontal asymptote: $y = 3$

Problem 10

a) $f(x) = 4 + \frac{2}{x-3}$ so vertical asymptote: $x = 3$, horizontal asymptote: $y = 4$
b) $f(x) = 20 - \frac{10}{2x-3}$ so vertical asymptote: $x = \frac{3}{2}$, horizontal asymptote: $y = 20$
c) Since $x^2 + 3$ is positive for all x, $f(x)$ is defined for all x, so no vertical asymptote. Horizontal asymptote: $y = 0$
d) $f(x) = 4 - \frac{4(3x-7)}{(x-1)(x-3)}$ so vertical asymptotes: $x = 1$ and $x = 3$, horizontal asymptote: $y = 4$
e) $f(x) = x + 10 + \frac{75}{x-7}$ so vertical asymptote: $x = 7$, non-vertical asymptote: $y = x + 10$
f) $f(x) = x + 10 + \frac{84}{x-8}$ so vertical asymptote: $x = 8$, non-vertical asymptote: $y = x + 10$

Problem 11

a) $f(x)$ has a zero between $x = 4$ and $x = 5$ by the intermediate value theorem because $f(4) = \sqrt{4-2} - 4 + 3 = 0.41 > 0$ while $f(5) = \sqrt{5-2} - 5 + 3 = -0.27 < 0$ and the function is defined and is continuous on the whole interval.

b) $f(x)$ has a zero between $x = 5$ and $x = 6$ by the intermediate value theorem because $f(5) = -0.80$ while $f(6) = 0.69 > 0$ and the function is defined and is continuous on the whole interval.
Note: $f(15) = -0.52 < 0$ together with $f(6) > 0$ tell that $f(x)$ has a zero between $x = 6$ and $x = 15$. So f(x) has at least 2 zeros on the interval $[5, 15]$.

c) $f(x) = \frac{2}{x-3}$ har no zeros on the interval $I = [2, 4]$ because the equation $\frac{2}{x-3} = 0$ has no solutions. Note: We can not use the intermediate value theorem even if $f(2) = -2 < 0$ and $f(4) = 2 > 0$ because $f(x)$ is not defined on the whole interval (even if $f(x)$ er continuous for all x where it is defined).