Key Problems

Problem 1.
Compute the definite integrals:

a) $\int_0^1 x \, \mathrm{d}x$ b) $\int_0^1 x^2 \, \mathrm{d}x$ c) $\int_0^1 x^3 \, \mathrm{d}x$ d) $\int_0^1 e^x \, \mathrm{d}x$ e) $\int_0^1 (e^x + e^{-x}) \, \mathrm{d}x$

f) $\int_{-1}^1 x \, \mathrm{d}x$ g) $\int_{-1}^1 x^2 \, \mathrm{d}x$ h) $\int_{-1}^1 x^3 \, \mathrm{d}x$ i) $\int_{-1}^1 e^x \, \mathrm{d}x$ j) $\int_{-1}^1 (e^x + e^{-x}) \, \mathrm{d}x$

Problem 2.
Compute the definite integrals:

a) $\int_0^1 xe^x \, \mathrm{d}x$ b) $\int_0^1 x \ln(x^2+1) \, \mathrm{d}x$ c) $\int_0^1 \frac{1}{x^2+5x+6} \, \mathrm{d}x$ d) $\int_0^1 \frac{1}{x^2+4x+4} \, \mathrm{d}x$

e) $\int_{-1}^1 xe^x \, \mathrm{d}x$ f) $\int_{-1}^1 x \ln(x^2+1) \, \mathrm{d}x$ g) $\int_{-1}^1 \frac{1}{x^2+5x+6} \, \mathrm{d}x$ h) $\int_{-1}^1 \frac{1}{x^2+4x+4} \, \mathrm{d}x$

Problem 3.
Compute the definite integral $\int_1^2 1/x \, \mathrm{d}x$. Then, for each value of n, estimate the area under the graph of $y = 1/x$ in the interval [1,2] with a Riemann sum using n subintervals:

a) $n = 2$ b) $n = 4$ c) $n = 8$

Problem 4.
Compute the definite integral $\int_0^2 (x^3 - 3x + 1) \, \mathrm{d}x$, and explain that the answer can be interpreted as $A_1 - A_2 + A_3$, where A_1, A_2, A_3 are the areas of three regions R_1, R_2, R_3 in the xy-plane bounded by the graph of $f(x) = x^3 - 3x + 1$ and the x-axis. Show these regions in a figure. A rough sketch is sufficient.

Problem 5.
Let R be the region bounded by the graph of $y = \ln(2 + x)$, the line $y = 2$, and the y-axis. Show the region R in a figure, and compute the area of R.

Problem 6.
Let R be the region bounded by the graphs of $y = x$ og $y = x^2$. Show the region R in a figure, and compute the area of R.

Problem 7.
Compute the integrals. Interpret each integral as an area, and show it in a figure.

a) $\int_1^\infty 1/x \, \mathrm{d}x$ b) $\int_1^\infty 1/x^2 \, \mathrm{d}x$ c) $\int_0^1 -\ln x \, \mathrm{d}x$ d) $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, \mathrm{d}x$ e) $\int_0^\infty \frac{e^{1-\sqrt{x}}}{\sqrt{x}} \, \mathrm{d}x$

Problem 8.

Find the area under the graph of $y = 1/x$ in the interval $I = [1,2]$, and use this to show that

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\right)=\ln(2)$$

Problem 9.

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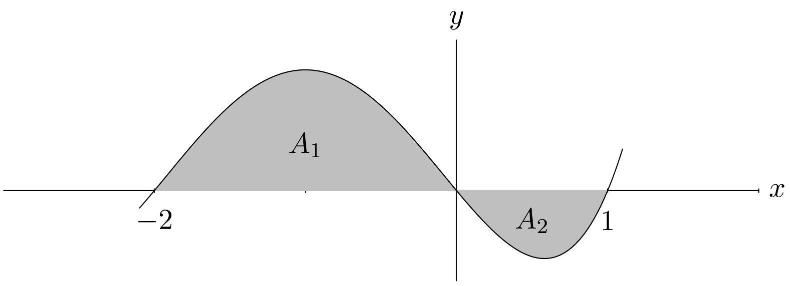
Compute these integrals:

a) **(6p)** $\displaystyle\int 30x\sqrt{x}\,\mathrm{d}x$ b) **(6p)** $\displaystyle\int xe^{-x}\,\mathrm{d}x$ c) **(6p)** $\displaystyle\int \frac{6-3x}{4-9x^2}\,\mathrm{d}x$



d) **(6p)** The graph of a function f is shown in the figure above. Determine the area A_1 when $A_2 = 22/15$ and

$$\int_{-2}^{1} f(x)\,\mathrm{d}x = \frac{18}{5}$$

Problem 10.

Optional: Problems from [Eriksen] (norwegian textbook)

Problem 5.6.1 - 5.6.5

Answers to Key Problems

Problem 1.

a) $1/2$	b) $1/3$	c) $1/4$	d) $e-1$	e) $e-1/e$
f) 0	g) $2/3$	h) 0	i) $e-1/e$	j) $2(e-1/e)$

Problem 2.

a) 1	b) $\ln(2)-1/2$	c) $2\ln(3)-3\ln(2)$	d) $1/6$
e) $2/e$	f) 0	g) $\ln(3)-\ln((2)$	h) $2/3$

Problem 3.

The definite integral $\int_1^2 1/x \, \mathrm{d}x = \ln(2) \approx 0.693$, and the approximations are given by:

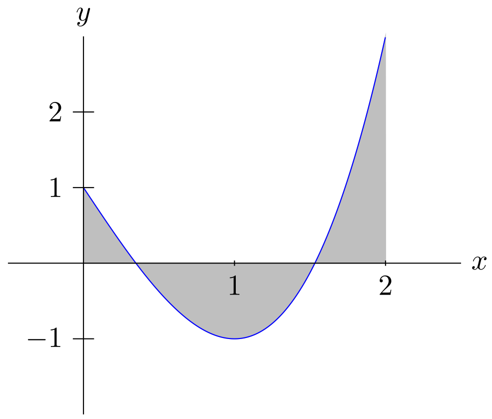
a) $0.5 \cdot 1/1 + 0.5 \cdot 2/3 \approx 0.833$

b) $0.25 \cdot 1/1 + 0.25 \cdot 4/5 + 0.25 \cdot 2/3 + 0.25 \cdot 4/7 \approx 0.760$

c) $0.125 \cdot 1 + 0.125 \cdot 8/9 + 0.125 \cdot 4/5 + 0.125 \cdot 8/11 + 0.125 \cdot 2/3 + 0.125 \cdot 8/13 + 0.125 \cdot 4/7 + 0.125 \cdot 8/15 \approx 0.725$

Problem 4.

The definite integral is $\int_0^2 (x^3 - 3x + 1) \, \mathrm{d}x = 0$. The graph is shown below, and we see that the regions R_1 and R_3 are above the x-axis while the region R_2 is below. Hence $A_1 - A_2 + A_3 = 0$.



Problem 5.

$e^2 - 6 + \ln(4)$

Problem 6.

$1/6$

Problem 7.

a) ∞ b) 1 c) 1 d) $2e - 2$ e) $2e$

Problem 8.

The area is $\ln(2)$, and the Riemann sum for this area using n subintervals is

$$\frac{1}{n} \cdot \frac{1}{1} + \frac{1}{n} \cdot \frac{1}{1 + 1/n} + \cdots + \frac{1}{n} \cdot \frac{1}{1 + (n-1)/n} = \frac{1}{n} + \frac{1}{n} \cdot \frac{n}{n+1} + \cdots + \frac{1}{n} \cdot \frac{n}{2n-1} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$$

The limit of this sum when $n \to \infty$ is therefore equal to the area $\ln(2)$.

Problem 9.

a) We write $x\sqrt{x} = x^{3/2}$ as a power, and use the power rule:

$$\int 30x\sqrt{x} \, \mathrm{d}x = \int 30x^{3/2} \, \mathrm{d}x = 30 \cdot \frac{2}{5} x^{5/2} + C = 12x^2\sqrt{x} + C$$

b) We use integration by parts with $u' = e^{-x}$ og $v = x$, which gives $u = -e^{-x}$ and $v' = 1$, and therefore

$$\int xe^{-x}\,\mathrm{d}x = -e^{-x}\cdot x - \int -e^{-x}\cdot 1\,\mathrm{d}x = -xe^{-x} + \int e^{-x}\,\mathrm{d}x = -xe^{-x} - e^{-x} + C$$

c) We factorize the denominator as $4 - 9x^2 = (2+3x)(2-3x)$, and simplify the expression in the integral using partial fractions. This gives

$$\frac{6-3x}{4-9x^2} = \frac{A}{2+3x} + \frac{B}{2-3x} \quad \Rightarrow \quad 6 - 3x = A(2-3x) + B(2+3x)$$

Hence $6 - 3x = (2A + 2B) + (3B - 3A)x$, or $2A + 2B = 6$ and $3B - 3A = -3$. The system of equations can be written $A + B = 3$ and $B - A = -1$. This gives $2B = 2$ by adding the equations, which means that $B = 1$ and $A = 2$. The integral is therefore

$$\int \frac{6-3x}{4-9x^2}\,\mathrm{d}x = \int \frac{2}{2+3x} + \frac{1}{2-3x}\,\mathrm{d}x = \frac{2}{3}\ln|2+3x| - \frac{1}{3}\ln|2-3x| + C$$

d) We divide the integral in two parts and express it in terms of the areas A_1 and A_2. This gives

$$\int_{-2}^{1} f(x)\,\mathrm{d}x = \int_{-2}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x = A_1 - A_2$$

since $f(x) > 0$ in the interval $(-2{,}0)$ and $f(x) < 0$ in the interval $(0{,}1)$. We solve for A_1, and find that

$$A_1 = \int_{-2}^{1} f(x)\,\mathrm{d}x + A_2 = \frac{18}{5} + \frac{22}{15} = \frac{76}{15} \approx 5.07$$