Key Problems

Problem 1.
Determine the number of solutions of the linear system for each value of the parameter a:

$$a)\quad \begin{array}{rcrcrcr} x & + & 3y & + & az & = & 0 \\ 2x & - & ay & + & 3z & = & 0 \\ 3x & + & 2y & + & 4z & = & 0 \end{array} \qquad b)\quad \begin{array}{rcrcrcr} 2x & + & ay & - & z & = & a-5 \\ -x & + & 2y & + & az & = & -3 \\ ax & - & y & + & 2z & = & a+10 \end{array}$$

Problem 2.
We start with a quadratic matrix A, perform an elementary row operation, and obtain a new matrix B. Is it always the case that $|A| = |B|$? Give reasons why/why not, and give examples.

Problem 3.
Determine when the system has exactly one solution, og use Kramer's rule to find the solutions in these cases:

$$a)\quad \begin{array}{rcrcr} x & + & ay & = & 3 \\ ax & + & 4y & = & 1 \end{array} \qquad b)\quad \begin{array}{rcrcr} ax & + & y & = & 1 \\ -x & + & ay & = & 2 \end{array}$$

Problem 4.
Determine the number of solutions of the linear system for each value of the parameter a, and find the solutions when the system is consistent:

$$a)\quad \begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ ax & + & 4y & + & 3z & = & 25 \\ ax & + & y & - & z & = & 12 \end{array} \qquad b)\quad \begin{array}{rcrcrcr} ax & + & y & + & z & = & 1 \\ x & + & ay & + & z & = & 2 \\ x & + & y & + & az & = & -3 \end{array}$$

Problem 5.
We consider the vectors given by

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

Draw the vectors in a 2-dimensional coordinate system. The compute the following vectors, and show them in the same coordinate system:

a) $\mathbf{u}+\mathbf{v}$ b) $\mathbf{v}+\mathbf{w}$ c) $\mathbf{v}-\mathbf{w}$ d) $2\mathbf{u}$ e) $-\mathbf{v}$ f) $3\mathbf{u}+\mathbf{w}$

Problem 6.
Solve the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ for the vectors below. Is $\mathbf{b}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \\ -8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Problem 7.
Solve the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ when the vectors are given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \\ a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Problem 8.
We consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{b}$ given below. Determine all (a,b,c,d) such that $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then use this to determine whether $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ when $(a,b,c,d) = (0,0,1,1)$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 7 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Problem 9.
You have 400.000 kr and want to invest in a portfolio of securities. You may choose a combination of the shares A, B, C with prices $p_A = 60$ kr, $p_B = 75$ kr and $p_C = 320$ kr per share at the time of the investment. We estimate that at a given time in the future, one of three scenarios will occur. The prices of the shares in each scenario are given in the table below. We write x, y, z for the number of shares you buy in the shares A, B, C. For simplicity

	Pris A	Pris B	Pris C
Initial price	60	75	320
Scenario 1	80	80	350
Scenario 2	100	25	500
Scenario 3	40	100	55

we assume that x,y,z can be any real numbers. Hence, we allow to buy a negative number of shares (short-selling) and to buy a non-integer number of shares.

a. We assume that the condition $60x + 75y + 320z = 400.000$ is met. What does it mean?

b. We write R_1, R_2 og R_3 for the return of the portfolio in each of the three scenarios. Is it possible to choose the portfolio such that $(R_1, R_2, R_3) = (50.000, 25.000, -100.000)$? If so, determine the portfolio that we have to buy.

c. Is it possible to choose a portfolio such that $R_1 > 0$ and $R_2 = R_3 = 0$? If so, which portfolio should we buy? Interpret the answer.

d. Describe all triples (R_1,R_2,R_3) of possible returns in the three scenarios. Are there are any portfolios such that $R_1, R_2, R_3 > 0$?

Problem 10.
Optional: Problems from [Eriksen] (norwegian textbook)
Problem 6.4.4 - 6.4.7, 6.5.2 - 6.5.3 (textbook) 9.19 - 9.22, 9.24, 9.26 (workbook)

Answers to Key Problems

Problem 1.
a) Infinitely many solutions for $a = \pm 1$, a unique solution for $a \neq \pm 1$
b) Infinitely many solutions for $a = -1$, a unique solution for $a \neq -1$

Problem 2.
If the operation $A \to B$ is to add a multiple of one row to another row, then $|A| = |B|$. If we switch two rows, then $|B| = -|A|$. If we multiply one row with $c \neq 0$, then $|B| = c \cdot |A|$.

Problem 3.
a) $(x,y) = \left(\dfrac{12-a}{4-a^2}, \dfrac{1-3a}{4-a^2} \right)$ for $a \neq \pm 2$ b) $(x,y) = \left(\dfrac{a-2}{a^2+1}, \dfrac{2a+1}{a^2+1} \right)$ for all a

Problem 4.

a) No solutions for $a = 7$, one solution $(x,y,z) = \left(\dfrac{17}{a-7}, \dfrac{-a-61}{a-7}, \dfrac{4a+23}{a-7} \right)$ for $a \neq 7$.

b) No solutions for $a = 1$, infinitely many solutions $(x,y,z) = (z - 4/3, z - 5/3, z)$ with z free for $a = -2$, and one solution for $a \neq 1, -2$ given by
$$(x,y,z) = \left(\frac{1}{a-1}, \frac{2}{a-1}, \frac{-3}{a-1} \right)$$

Problem 5.
a) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ b) $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ c) $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$ d) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ e) $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ f) $\begin{pmatrix} 2 \\ 11 \end{pmatrix}$

Problem 6.
The general solution is $(x,y,z) = (-4z-1, z+1, z)$ with z free. A particular solution is given by (for example) letting $z = 0$, which gives $(-1,1,0)$, and this means that $\mathbf{b} = -1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 7.
For $a = -8$, there are infinitely many solutions $(x,y,z) = (-4z-1, z+1, z)$ with z free. For $a \neq -8$, there is exactly one solution $(x,y,z) = (-1,1,0)$.

Problem 8.
It is a linear combination if and only if $-7a + 9b - 5c + 3d = 0$, and $(a,b,c,d) = (0,0,1,1)$ does not satisfy this equation. Hence $\mathbf{b}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in this case.

Problem 9.

a. This is a budget condition (the total cost of the shares we buy is 400.000 kr).

b. Yes, we may choose $(x,y,z) = (1187\,{}^1\!/_2, 2250, 500)$.

c. Yes, if $R_1 = 80.000$. We may choose $(x,y,z) = (3333\,{}^1\!/_3, 2666\,{}^2\!/_3, 0)$. This means that we may invest without risking to lose money, and with positive expected return (a very fortunate situation for us!)

d. The possible (R_1,R_2,R_3) satisfy the equation $5R_1 - 2R_2 - 2R_3 = 400.000$. We may choose $R_1, R_2, R_3 > 0$ (a positive return in all possible scenarios), for example $R_1 = R_2 = R_3 = 400.000$.