## Key Problems

**Problem 1.**
We consider the function $f(x,y) = x^2 - 2x + 4y^2$.

a) Show that the level curve $f(x,y) = c$ is an ellipse when $c > -1$, and determine its center $(x_0,y_0)$ and its half-axes $a$ and $b$. Use this to sketch the level curves for $c = 0, 1, 2, 3$ in the same coordinate system.
b) Find the tangent to the level curve at $(x,y) = (1,1)$ and at $(x,y) = (2,1/2)$. Show the tangents in the figure.
c) Find $\nabla f(1,1)$ and $\nabla f(2,1/2)$, and show them in the figure. What happens to $f(x,y)$ along the gradients?
d) Does it seem like the function $f$ has a minimum or maximum value? Explain why/why not.

**Problem 2.**
Find the partial derivatives $f'_x$ and $f'_y$:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

**Problem 3.**
Find the Hessian $H(f)$, and compute $H(f)(1,1)$:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

**Problem 4.**
Find the gradient $\nabla f(1,1)$ of $f$ at $(1,1)$, and use this to find the directional derivative $f'_{\mathbf{a}}(1,1)$ of $f(x,y)$ at $(1,1)$ along the vector $\mathbf{a} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}^T$:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3)$

**Problem 5.**
Determine the tangent of $f(x,y) = c$ at $(x,y) = (1,1)$:

a) $f(x,y) = 2x + 3y,\ c = 5$
b) $f(x,y) = x^2 + y^2,\ c = 2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2,\ c = 7$
d) $f(x,y) = x^2 - 2x + 4y^2,\ c = 3$
e) $f(x,y) = x^3 - 3xy + y^3,\ c = -1$
f) $f(x,y) = y^2 - x^3 + 3x,\ c = 3$
g) $f(x,y) = \sqrt{x^2 + y^2},\ c = \sqrt{2}$
h) $f(x,y) = \ln(x^2y^2 - x^2 - y^2 + 3),\ c = \ln 2$

**Problem 6.**
Show that the gradient $\nabla f(a,b)$ is normal to the tangent of the level curve $f(x,y) = c$ at the point $(a,b)$, and that $f$ increases when we move along the direction of the gradient.

**Problem 7.**
Problem 7.1.1 - 7.1.4, 7.2.1 - 7.2.2, 7.3.1 - 7.3.2 (norwegian textbook, optional)

## Answers to Key Problems

**Problem 1.**

a) Ellipses with center (1,0) and with half-axes $a = \sqrt{c+1}$ and $b = \sqrt{c+1}/2$.

b) The tangents have equations $y = 1$ and $y = -x/2 + 3/2$.

c) $\nabla f(1{,}1) = \begin{pmatrix}0 & 8\end{pmatrix}^T$, $\nabla f(2{,}1/2) = \begin{pmatrix}2 & 4\end{pmatrix}^T$, and the function increases along the gradient.

d) No maximum value (the half-axes increase when $c$ increases). The minimum value is $f(1{,}0) = -1$.

**Problem 2.**

a) $f'_x = 2,\ f'_y = 3$

b) $f'_x = 2x,\ f'_y = 2y$

c) $f'_x = 8x - 6y,\ f'_y = -6x + 18y$

d) $f'_x = 2x - 2,\ f'_y = 8y$

e) $f'_x = 3x^2 - 3y,\ f'_y = -3x + 3y^2$

f) $f'_x = -3x^2 + 3,\ f'_y = 2y$

g) $f'_x = \dfrac{x}{\sqrt{x^2+y^2}},\ f'_y = \dfrac{y}{\sqrt{x^2+y^2}}$

h) $f'_x = \dfrac{2x(y^2-1)}{x^2y^2 - x^2 - y^2 + 3},\ f'_y = \dfrac{2y(x^2-1)}{x^2y^2 - x^2 - y^2 + 3}$

**Problem 3.**

a) $H(f) = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

b) $H(f) = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

c) $H(f) = \begin{pmatrix}8 & -6\\ -6 & 18\end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix}8 & -6\\ -6 & 18\end{pmatrix}$

d) $H(f) = \begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}$

e) $H(f) = \begin{pmatrix}6x & -3\\ -3 & 6y\end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix}6 & -3\\ -3 & 6\end{pmatrix}$

f) $H(f) = \begin{pmatrix}-6x & 0\\ 0 & 2\end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix}-6 & 0\\ 0 & 2\end{pmatrix}$

g) $H(f) = (x^2+y^2)^{-3/2} \cdot \begin{pmatrix}y^2 & -xy\\ -xy & x^2\end{pmatrix},\quad H(f)(1{,}1) = \begin{pmatrix}\sqrt{2}/4 & -\sqrt{2}/4\\ -\sqrt{2}/4 & \sqrt{2}/4\end{pmatrix}$

h) $H(f) = (x^2y^2 - x^2 - y^2 + 3)^{-2} \cdot \begin{pmatrix}2(y^2-1)(-x^2y^2 + x^2 - y^2 + 3) & 8xy\\ 8xy & 2(x^2-1)(-x^2y^2 - x^2 + y^2 + 3)\end{pmatrix},\quad \begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$

**Problem 4.**

a) $\nabla f(1{,}1) = \begin{pmatrix}2 & 3\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 2a_1 + 3a_2$

b) $\nabla f(1{,}1) = \begin{pmatrix}2 & 2\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 2a_1 + 2a_2$

c) $\nabla f(1{,}1) = \begin{pmatrix}2 & 12\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 2a_1 + 12a_2$

d) $\nabla f(1{,}1) = \begin{pmatrix}0 & 8\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 8a_2$

e) $\nabla f(1{,}1) = \begin{pmatrix}0 & 0\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 0$

f) $\nabla f(1{,}1) = \begin{pmatrix}0 & 2\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 2a_2$

g) $\nabla f(1{,}1) = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = (a_1 + a_2)/\sqrt{2}$

h) $\nabla f(1{,}1) = \begin{pmatrix}0 & 0\end{pmatrix}^T,\ f'_{\mathbf{a}}(1{,}1) = 0$

**Problem 5.**

a) $y = -2x/3 + 5/3$

b) $y = -x + 2$

c) $y = -x/6 + 7/6$

d) $y = 1$

e) No tangent

f) $y = 1$

g) $y = -x + 2$

h) No tangent