

- Plan
1. Inverse functions
  2. Exponential functions
  3. Logarithms

1. Inverse functions

Ex  $f(x) = (x-3)^2$   
with domain  $D_f = [3, \rightarrow)$   
(so  $x \geq 3$ )

what is a function?

- an expression
- a table of function values
- a graph
- a situation

Table of function values

x	3	4	5	6	7	...	g(x)
f(x)	0	1	4	9	16	...	x

← the inverse function of  $f(x)$

so  $g(0) = 3$  ,  $g(1) = 4$  ,  $g(4) = 5$  .....

$f(g(0)) = f(3) = 0$

$f(g(1)) = f(4) = 1$

$f(g(4)) = f(5) = 4$

$g(f(3)) = g(0) = 3$

and  $g(f(4)) = g(1) = 4$

$g(f(5)) = g(4) = 5$

Definition

$f(x)$  with domain  $D_f$

and

$g(x)$  with domain  $D_g$

are inverse functions iff

$f(g(x)) = x$

and

$g(f(x)) = x$

for all  $x$  in  $D_g$

for all  $x$  in  $D_f$

Here the domain of  $g(x)$  is the range of  $f(x)$ . Short:  $D_g = R_f$

Also:  $f(x)$  is the inverse function of  $g(x)$

so  $R_g = D_f$

How to find an expression for the inverse function?

- ① Solve the equation  $y = f(x)$  for  $x$ .
- ② Switch the variables  $x$  and  $y$ .
- ③ Put  $D_g = R_f$  and determine  $R_f$ .

Ex  $f(x) = (x-3)^2$  with  $D_f = [3, \rightarrow)$   
We find  $g(x)$  with  $D_g$  by ① - ③.

- ① We solve the equation  
 $y = (x-3)^2$  for  $x$

- take the square root on each side

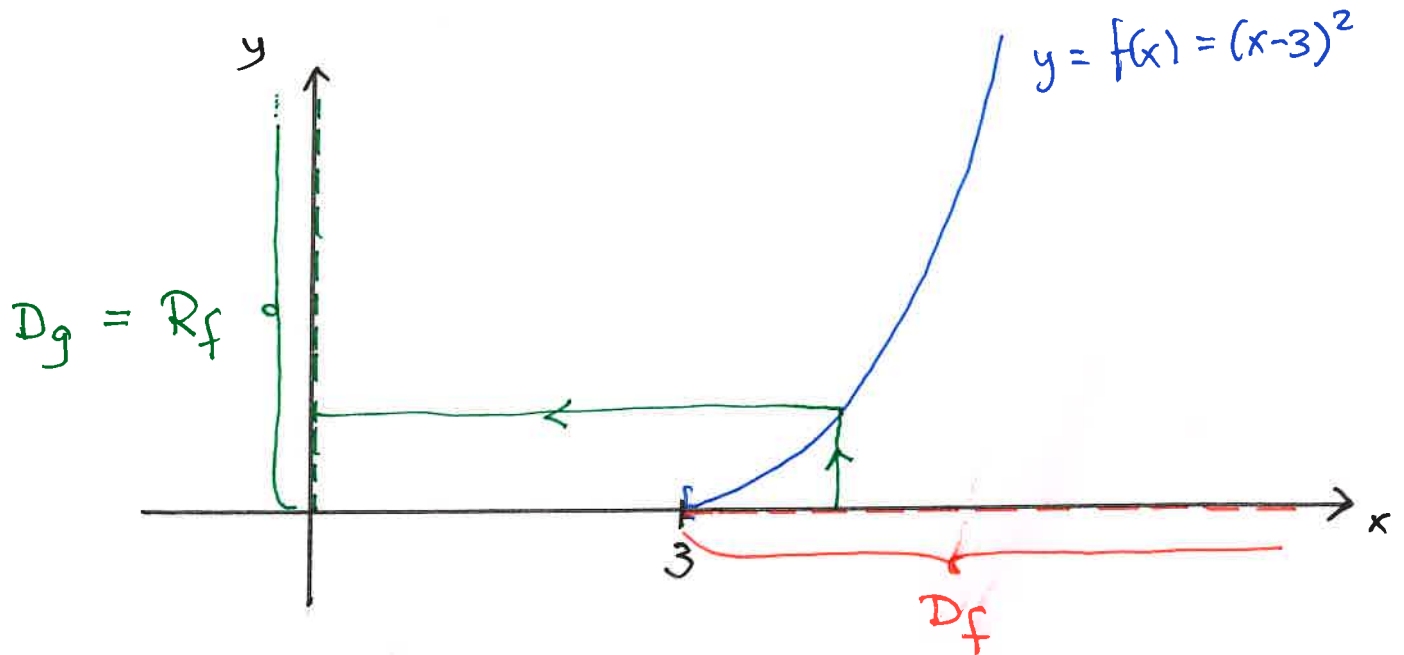
$$\sqrt{y} = |x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ -(x-3) & \text{if } x < 3 \end{cases}$$

so  $\sqrt{y} = x-3$  since  $x \in D_f = [3, \rightarrow)$   
( $x \geq 3$ ).

and then  $x = \underline{3 + \sqrt{y}}$

- ② Switch variables:  $y = g(x) = 3 + \sqrt{x}$

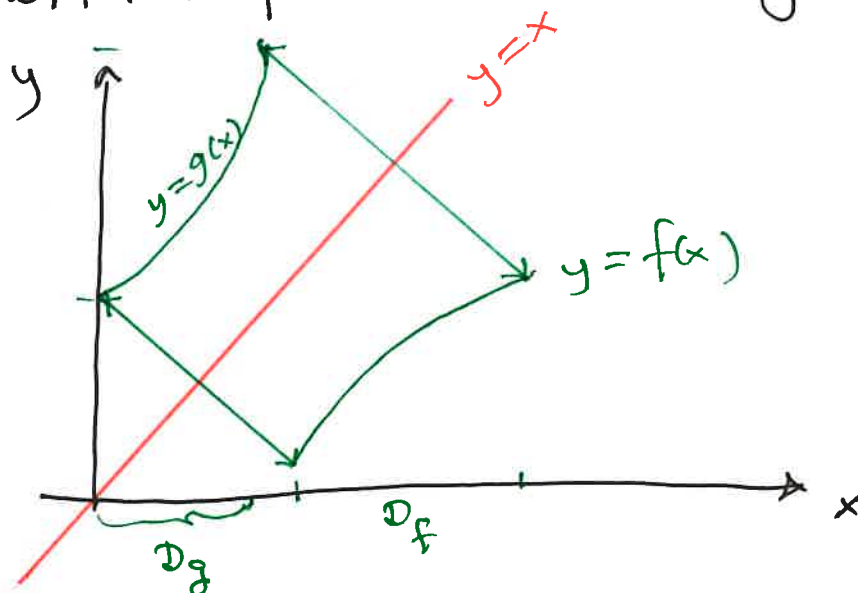
③  $D_g = R_f = [0, \rightarrow)$  because  
 $f(x) = (x-3)^2 = y$  has a solution  $x$   
 with  $x \geq 3$  for all values  $y \geq 0$



Note that  $f(g(x)) = ((3 + \sqrt{x}) - 3)^2 = x$   
 and  $g(f(x)) = 3 + \sqrt{(x-3)^2} = x$  (since  $x \geq 3$ )

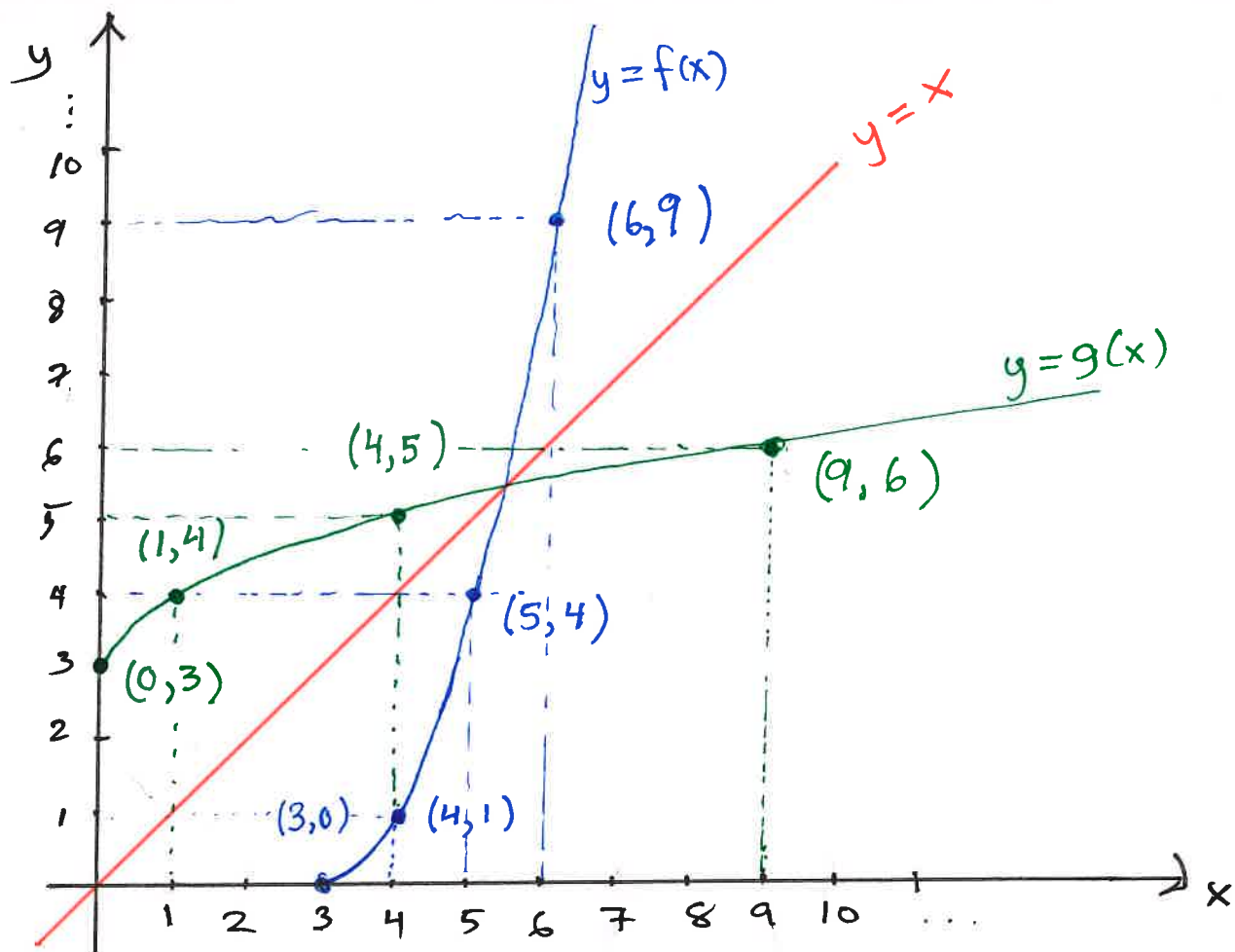
The graph of the inverse function

- is the mirror image of graph of  $f(x)$   
 with respect to the "diagonal"  $y = x$



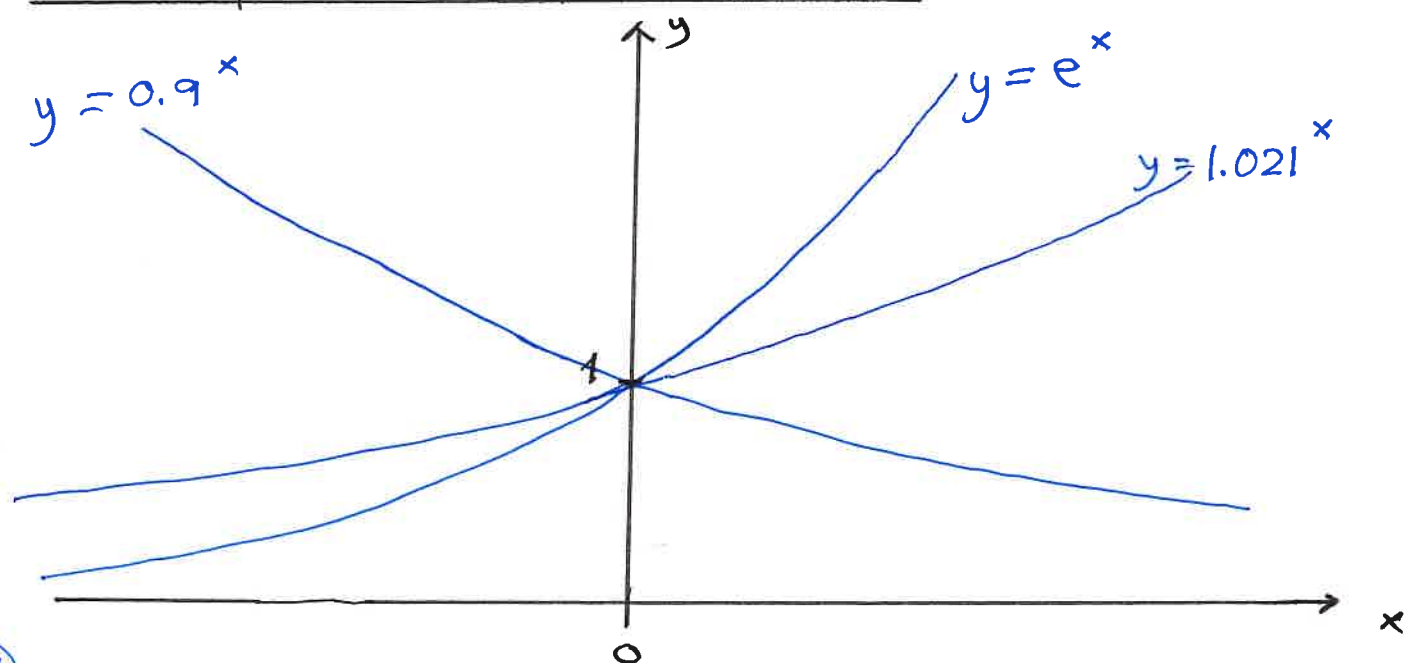
Ex  $f(x) = (x-3)^2$  with  $D_f = [3, \rightarrow)$

$x$	3	4	5	6	7	...	$g(x)$
$f(x)$	0	1	4	9	16	...	$x$



Start 11.05

## 2. Exponential functions



$a > 1$   $f(x) = a^x$  is strictly increasing  
without bounds

$$\text{and } a^x \xrightarrow{x \rightarrow -\infty} 0^+$$

$0 < a < 1$   $f(x) = a^x$  is strictly decreasing

$$\text{and } a^x \xrightarrow{x \rightarrow \infty} 0^+$$

Both cases  $D_f =$  all numbers on the number line

$$\text{and } R_f = \langle 0, \rightarrow \rangle$$

Power rules If  $f(x) = a^x$  then

$$f(x) \cdot f(y) = a^x \cdot a^y = a^{x+y} = f(x+y)$$

$$\text{and } \frac{1}{f(x)} = \frac{1}{a^x} = (a^{-1})^x = a^{-x} = f(-x)$$

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3. Logarithms Suppose  $a > 0$  and  $a \neq 1$   
Then  $g(x) = \log_a(x)$  is the inverse function  
of  $f(x) = a^x$  and

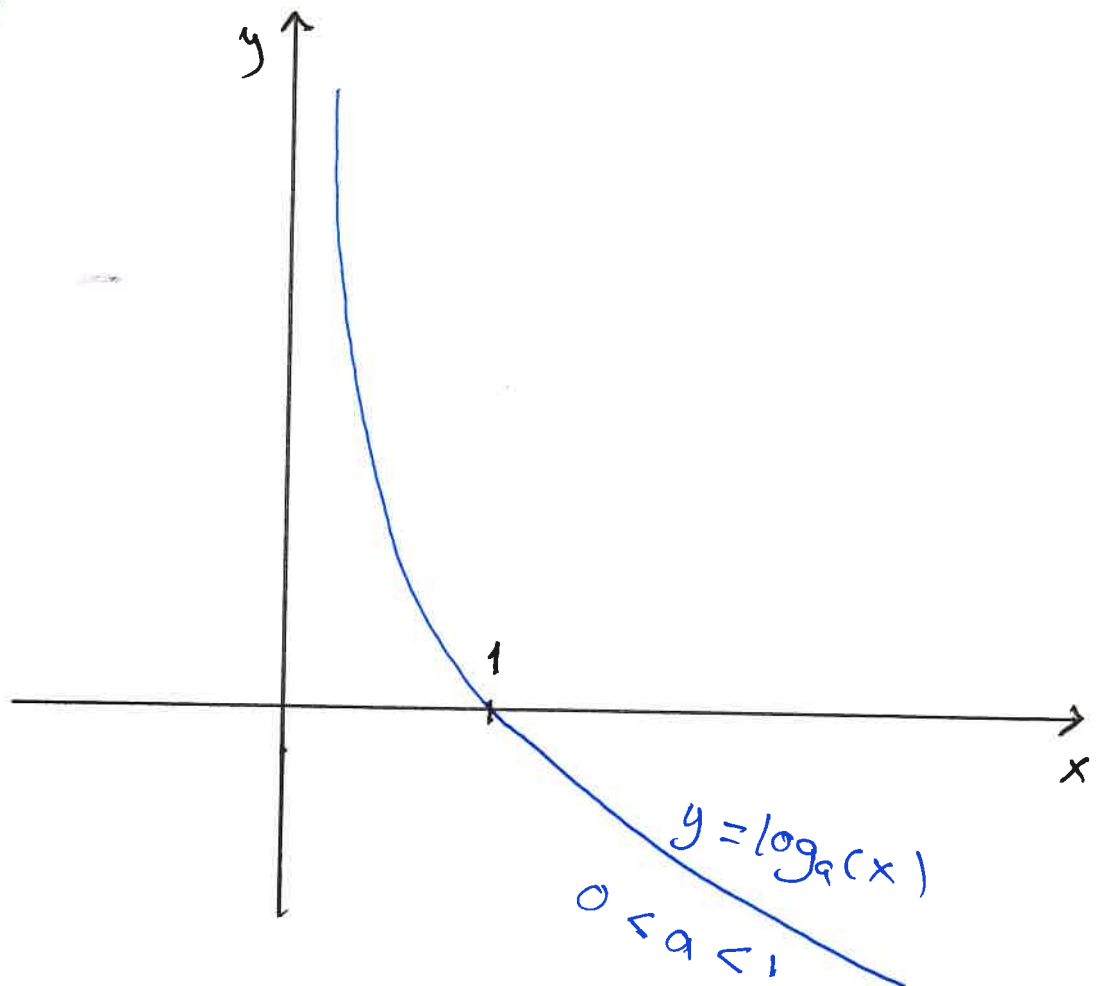
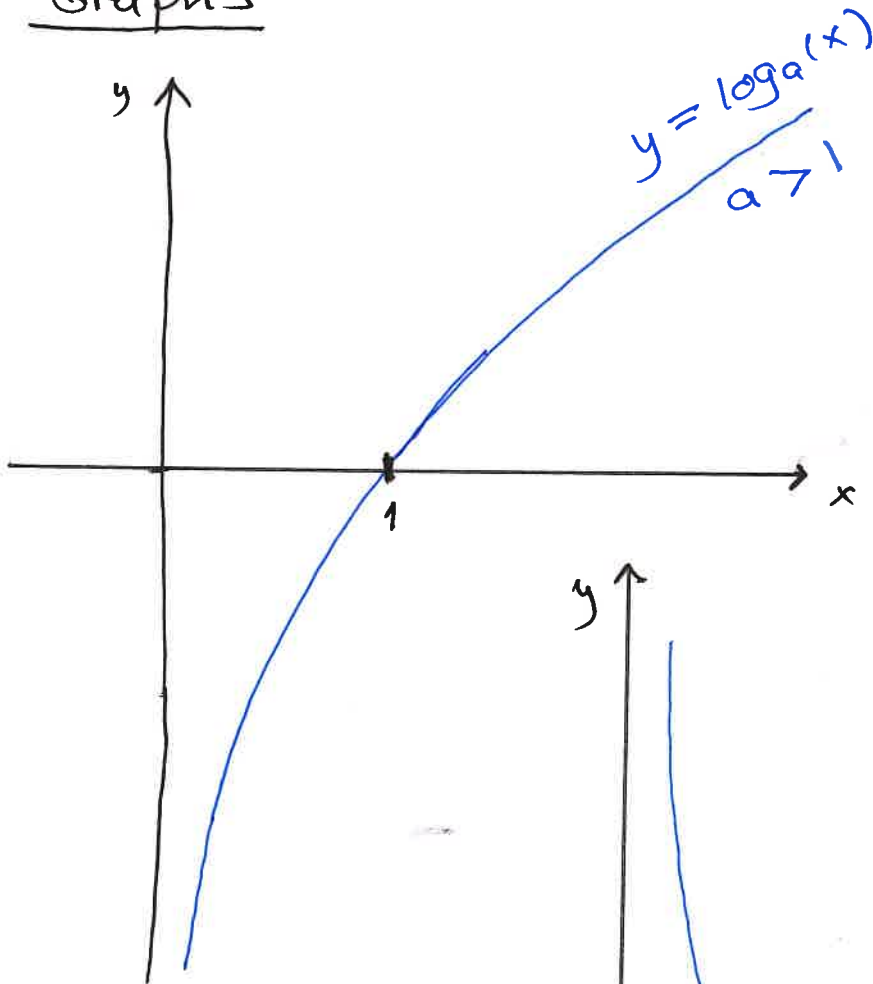
$$D_g = R_f = \langle 0, \rightarrow \rangle$$

Ex  $a=2$  ,  $\log_2(10) =$  the number which 2 has to be raised to to give 10

$$2^{3.322} \approx 10$$

$$\text{so } \log_2(10) \approx 3.322$$

### Graphs



Rules  $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^r) = r \cdot \log_a(x)$$

Definition  $\ln(x) = \log_e(x)$   $e = \text{Euler number}$

- is called the natural logarithm

$\ln(x)$  is the inverse function of  $e^x$ .

so  $e^{\ln(x)} = x$  and  $\ln(e^x) = x$