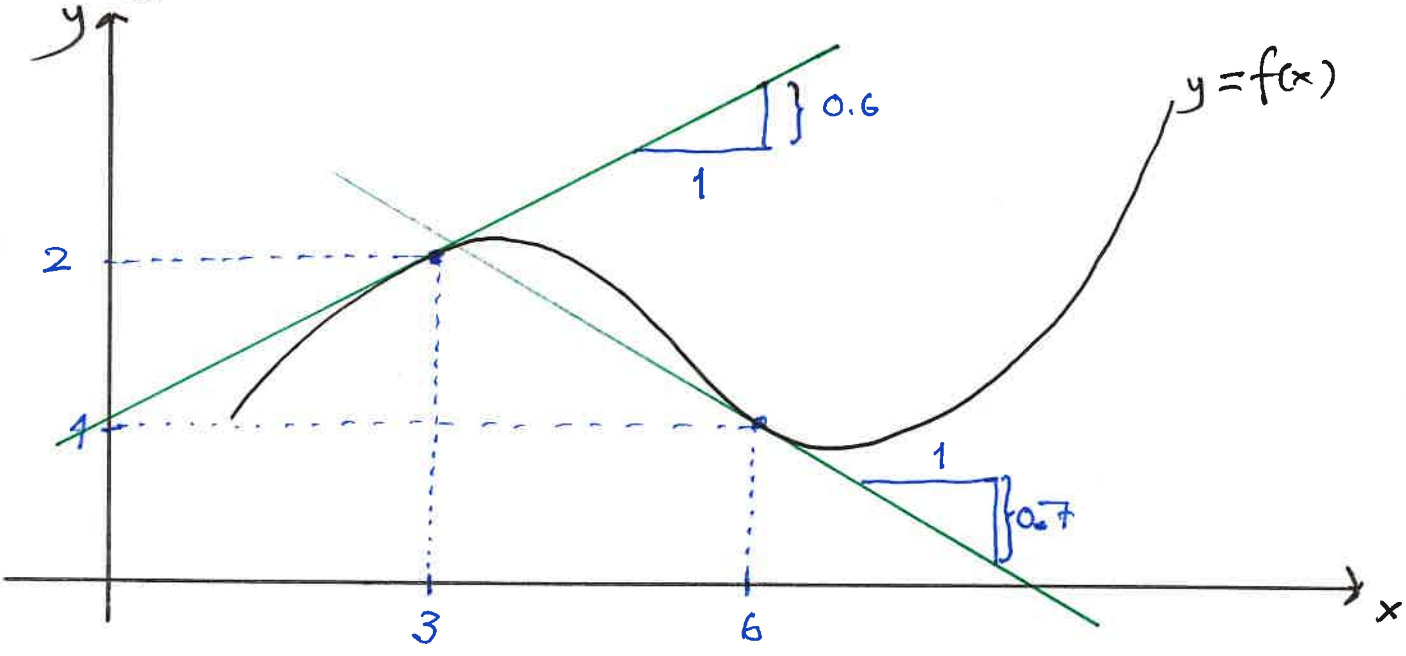
EBA 2911, lecture 11, 27 Oct. 2021, Runar Ile

Plan
1. Tangents and the derivative
2. The derivative as a function
3. Rules for differentiation

1. Tangents and the derivative



The tangent of the graph of $f(x)$ at the point $(3, 2)$ has slope 0.6

We write $f'(3) = 0.6$

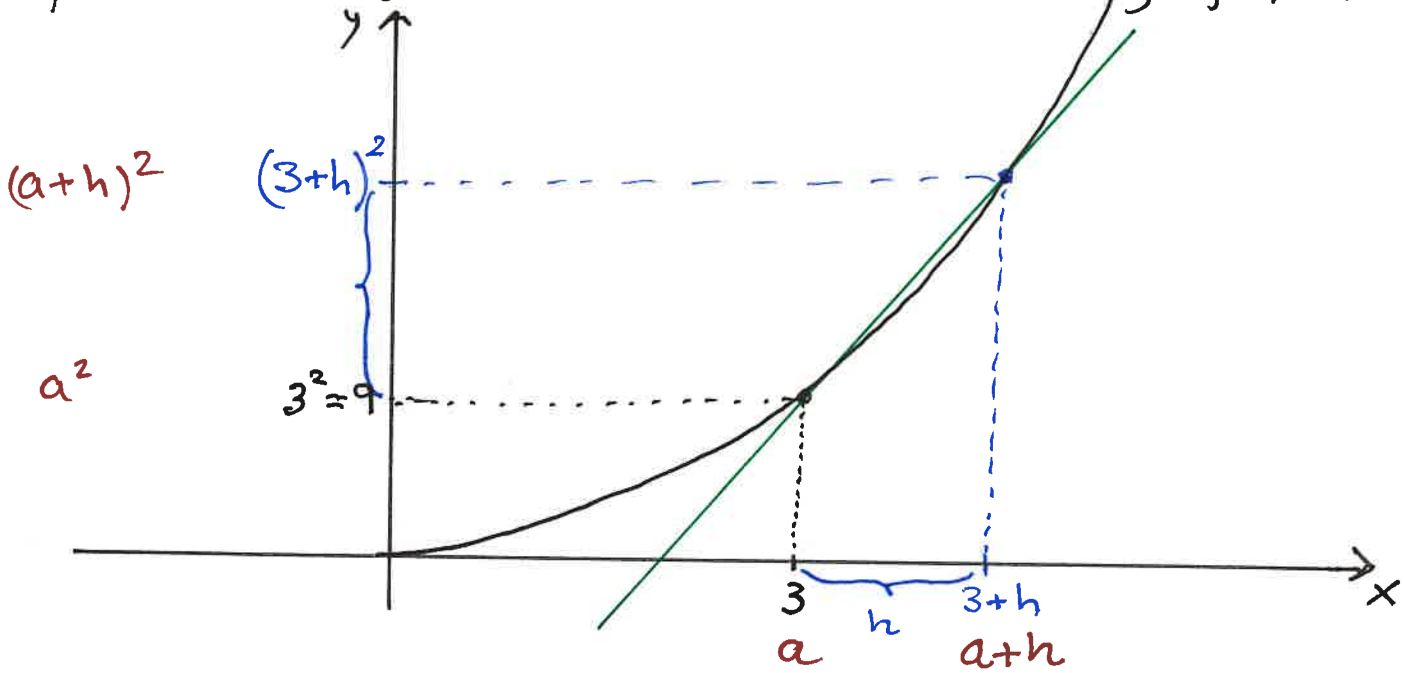
The tangent of the graph of $f(x)$ at the point $(6, 1)$ has slope -0.7

We write $f'(6) = -0.7$

Two important applications
1) To determine where the function increases/decreases and max/min
2) Approximate complicated functions with linear functions
 – typical for economic models

How to find the slope of the tangent?

Ex $f(x) = x^2$ and $(3, 9)$. What is the slope of the tangent?



The slope of this secant line is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\overset{(a+h)^2 - a^2}{(3+h)^2 - 3^2}}{h} = \frac{\overset{(a+h)(a+h) - a^2}{(3+h)(3+h) - 9}}{h}$$

$$= \frac{\overset{\cancel{a^2} + 2\cdot ah + h^2 - \cancel{a^2}}{\cancel{9} + 2\cdot 3\cdot h + h^2 - \cancel{9}}}{h} = \frac{\overset{2ah + h^2}{6h + h^2}}{h} = \frac{\overset{h(2a+h)}{\cancel{h}(6+h)}}{\cancel{h}}$$

$$2a + h \xrightarrow[h \to 0]{} 2a$$

$$= 6 + h \xrightarrow[h \to 0]{} 6$$ which has to be the slope of the tangent line to $f(x)$ through $(3, 9)$.

We write $f'(3) = 6$

also $f'(a) = 2a$

2. The derivative as a function

In the example: If $x = a$ then $f'(a) = 2a$

– this is a function, and we use x as variable: $f'(x) = 2x$

E.g. The slope of the tangent of $f(x)$ at $(-3, 9)$ is $f'(-3) = 2 \cdot (-3) = -6$

We could do the same with $f(x) = x^3$. We would (after similar calc.) get $f'(x) = 3x^2$

3. Rules of differentiation

Power rule $\quad f(x) = x^n$ gives $f'(x) = n \cdot x^{n-1}$ for all n.

Ex $\quad f(x) = x^{10}$, $\quad f'(x) = 10 \cdot x^9 \quad (n = 10)$

Ex $\quad f(x) = \sqrt[3]{x}$, $\quad f'(x) = \frac{1}{3} \cdot x^{\frac{1}{3}-1} \quad (n = \frac{1}{3})$

$= x^{\frac{1}{3}}$ $\quad = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

$$= \frac{1}{3} \cdot \frac{1}{x^{2/3}} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}$$

$$= \frac{1}{3 \cdot \sqrt[3]{x^2}}$$

Start: 11.00

The sum rule   If $f(x) = g(x) + h(x)$

then $f'(x) = g'(x) + h'(x)$

Ex $f(x) = x + x^3$, then $f'(x) = 1 + 3x^2$

The constant rule   If k is a constant number

and $f(x) = k \cdot g(x)$ then

$f'(x) = k \cdot g'(x)$

Ex $k=7$, $g(x) = x^2$, then $f(x) = 7x^2$

and $f'(x) = 7 . 2x = 14x$

The product rule   If $f(x) = g(x) \cdot h(x)$

then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

Ex $f(x) = (5x^3 - 2x + 1) \cdot (3x + 7)$

Calculate the $f'(x)$ by using the product rule.

$g(x) = 5x^3 - 2x + 1$ $h(x) = 3x + 7$

$g'(x) = 15x^2 - 2$ $h'(x) = 3$

so $f'(x) = (15x^2 - 2) \cdot (3x + 7) + (5x^3 - 2x + 1) \cdot (3)$

note the parentheses

calculate

$= \underline{\underline{60x^3 + 105x^2 - 12x - 11}}$

The quotient rule Suppose $f(x) = \frac{g(x)}{h(x)}$

Then $$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Ex $f(x) = \frac{3x+1}{2x+5}$. Then

$g(x) = 3x+1$ and $h(x) = 2x+5$

$g'(x) = 3$ $h'(x) = 2$

note the parentheses!

$$f'(x) = \frac{3 \cdot (2x+5) - (3x+1) \cdot 2}{(2x+5)^2}$$

note the sign
- for the whole parenthesis

$$= \frac{3 \cdot 2x + 3 \cdot 5 - (3x \cdot 2 + 1 \cdot 2)}{(2x+5)^2}$$

note this — !!

$$= \frac{6x + 15 - 6x - 2}{}$$

$$= \frac{13}{(2x+5)^2}$$ ← usually better not to expand denominator

The chain rule

the inner function

If $f(x) = g(\overbrace{u(x)})$

the outer function

Then $f'(x) = g'(u) \cdot u'(x)$ where $u = u(x)$

Ex $f(x) = (x^2+2)^{10}$

Put $u = u(x) = x^2+2$ and $g(u) = u^{10}$

$u'(x) = 2x$ $\qquad g'(u) = 10u^9$

Then $f'(x) = 10u^9 \cdot 2x$

$= 10 \cdot (x^2+2)^9 \cdot 2x = \underline{\underline{20x \cdot (x^2+2)^9}}$

Two functions

$f(x) = e^x$ and $g(x) = \ln(x)$

$f'(x) = e^x$ $\qquad g'(x) = \frac{1}{x} = x^{-1}$

Ex $f(x) = e^{3x}$

$u(x) = 3x$ and $g(u) = e^u$

$u'(x) = 3$ $\qquad g'(u) = e^u$

so $f'(x) = e^u \cdot 3$

$= \underline{\underline{3e^{3x}}}$

Ex $f(x) = \ln(x^2+1)$

$f'(x) = \frac{2x}{x^2+1}$

by the chain rule with $u = x^2+1$ and $g(u) = \ln(u)$.