Plan

About the course:

find links in the web page

- Lectures Wed 10-12
- Problem sessions Thu 12-15
- Later: video (1h), problems from the problem set.

Padlet: link in the web page
password: 2910

Topics:

1. Integration
2. Matrix and vector computations
3. Function in two variables

+ everything from last semester

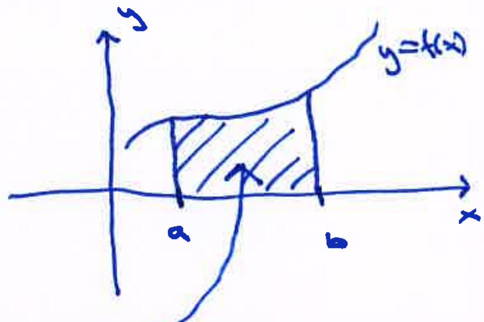
Exams:

- course paper EBA29103
- final exam EBA29104

+ retake exams

1 Definite integrals

Defn: $\int_a^b f(x)\,dx = \left\{ \begin{array}{l} \text{area of the} \\ \text{region under} \\ \text{the graph of} \\ f \text{ in } [a,b] \end{array} \right\}$



Assumptions:

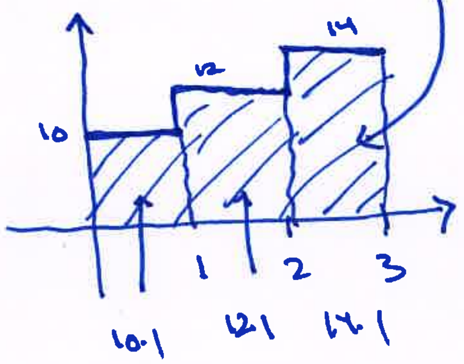
i) $f(x)$ is a cont. fn. on $[a,b]$

ii) $f(x) \geq 0$ on $[a,b]$

iii) $a < b$

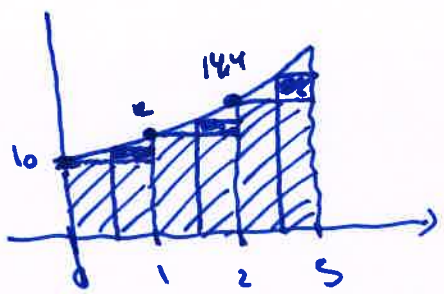
Ex: Rental income over three years:

$10 + 12 + 14 = 36$



Continuous change in the rent:

$f(x) = 10 \cdot 1{,}2^x$



Total income in the three years

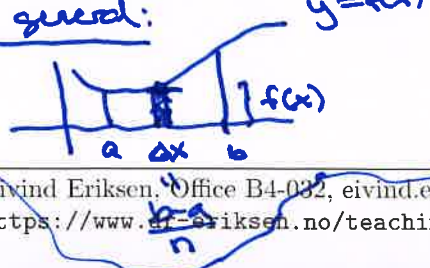
$= \int_0^3 10 \cdot 1{,}2^x\,dx =$ area under the graph in $[0,3]$

$\simeq 10 + 12 + 14{,}4 = 36.4$

$\underset{\text{approx. (Riemann sum)}}{\shortparallel}$

$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

In general:



rectangular box has area $f(x)\cdot\Delta x \to f(x)\cdot dx$

(2) Anti-derivatives and indefinite integrals

Defn: An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Ex: $f(x) = 2x \xrightarrow[\text{antiderivative}]{} F(x) = x^2$ since $(x^2)' = 2x$

or $F(x) = x^2+1$ since $(x^2+1)' = 2x$

or $F(x) = x^2 + C$ since $(x^2+C)' = 2x$

Fact: If $f(x)$ has antiderivative $F(x)$, then any other antiderivative can be written $F(x)+C$ where C is a constant.

Defn: The indefinite integral of a fn. $f(x)$ is

$$\int f(x)\,dx = F(x) + C \leftarrow \text{all antiderivatives of } f(x)$$

where $F'(x) = f(x)$.

Ex: $\int 2x\,dx = \underline{\underline{x^2 + C}}$

C: integration constant

Formal symbols: $\int$ = integration sign

dx: x is the integration variable

③ Integration rules: How to compute $\int f(x)dx$

Ex: $\int \underbrace{3+x+x^2}\,dx = \underline{3x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + C}$

Integral rules:

i) Power rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

iii) $\int u(x) \pm v(x)\,dx = \int u(x)\,dx \pm \int v(x)\,dx$

iv) $\int c \cdot u(x)\,dx = c \cdot \int u(x)\,dx \quad$ (c const.)

v) Exponentials: $\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C \quad (a>0)$

Ex: $\int 3x^5\,dx = 3\int x^5\,dx = 3\left(\frac{x^6}{6}\right) + C = \underline{\frac{1}{2}x^6 + C}$

$\int 3x^5 + 6x^{12}\,dx = \int 3x^5\,dx + \int 6x^{12}\,dx$

$= \frac{1}{2}x^6 + 6 \cdot \frac{x^{13}}{13} + C$

$= \underline{\frac{1}{2}x^6 + \frac{6}{13}x^{13} + C}$

$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C$

$= \frac{2}{3}x^{3/2} + C = \underline{\frac{2}{3}x\sqrt{x} + C}$

$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = \underline{-\frac{1}{x} + C}$

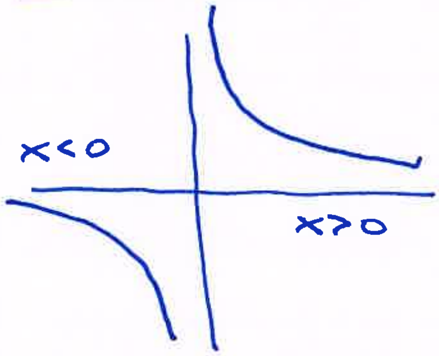
Ex:

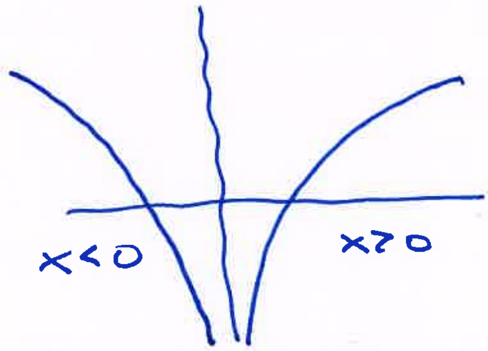
$$\int \frac{x^2-2x+3}{x}\,dx = \int \frac{x^2}{x} - \frac{2x}{x} + \frac{3}{x}\,dx$$

$$= \int x - 2 + 3\cdot\frac{1}{x}\,dx = \frac{1}{2}x^2 - 2x + 3(\ln|x|) + C$$

$$= \underline{\underline{\frac{1}{2}x^2 - 2x + 3\ln|x| + C}}$$

Explanation: $\int \frac{1}{x}\,dx = \ln|x| + C$





$f(x) = \frac{1}{x}\ ,\ x \neq 0$

$F(x) = \ln(x)\ ,\ x > 0$

$F(x) = \ln(-x)\ ,\ x < 0$

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) + C\ , & x > 0 \\ \ln(-x) + C\ , & x < 0 \end{cases}$$

$$\boxed{\big(\ln(-x)\big)' = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x} \quad \text{ok}}$$

$$= \underline{\underline{\ln|x| + C}}\ ,\ x \neq 0$$

④ Substitution:

More advanced integration techniques:
i) substitution
ii) integration by parts
iii) partial fractions

Ex: $\int e^{2x}\,dx = \int e^{u}\,dx$

$u = 2x$
$du = u' \cdot dx$
$du = 2\,dx$
$\frac{1}{2}\,du = dx$

We know that
$\int e^x\,dx = e^x + C$

General formula for substitution:
$\boxed{du = u' \cdot dx}$
$dx = \frac{1}{u'}\,du$

Ex (cont.d)
$= \int e^u \cdot \frac{1}{2}\,du = \frac{1}{2}\int e^u\,du = \frac{1}{2}(e^u) + C$
$= \frac{1}{2}e^{2x} + C$

note:
$(e^{2x})' = e^{2x} \cdot 2$
$\int e^{2x}\,dx = \frac{e^{2x}}{2} + C$

Ex: $\int x\sqrt{x^2+1}\,dx = \int x\sqrt{u}\,dx = \int \cancel{x}\cdot\sqrt{u}\cdot\frac{1}{2\cancel{x}}\,du$

$\boxed{\begin{array}{l} u = x^2+1 \\ du = u' \cdot dx = 2x\,dx \end{array}} \rightarrow dx = \frac{1}{2x}\,du$

$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\,u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C$

Note: We choose $u = x^2+1$, use $du = u'\,dx$, and transform the integral to one in the form $\int g(u)\,du$ (with u and du, no x or dx). Then we try to solve that integral, using u as a integration variable.