Plan

1 Substitution
2 Integration by parts
3 Partial fractions

Recall:
- integration rules
- substitution

① Substitution

$$\int f(x)\,dx \overset{\text{substitution}}{=} \int g(u)\,du \rightarrow \text{solve this integral}$$

$$\boxed{\begin{aligned} u &= u(x) \\ du &= u'(x)\,dx \end{aligned}}$$

$$\downarrow$$

$$dx = \frac{1}{u'(x)}\,du$$

Ex:

$$\int x e^{-x^2}\,dx = \int \cancel{x} e^u \cdot \left(-\frac{1}{2\cancel{x}}\right) du = \int -\frac{1}{2} e^u\,du$$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$dx = \frac{1}{-2x}\,du$$

$$= -\frac{1}{2} e^u + C$$

$$= -\frac{1}{2} e^{-x^2} + C$$

$$\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}} \cdot \cancel{x}\,du = \int u\,du$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$\downarrow$$

$$dx = x\,du$$

$$= \frac{1}{2} u^2 + C$$

$$= \frac{1}{2}(\ln x)^2 + C$$

Ex:

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^{1-\sqrt{x}}}{u}\cdot 2\sqrt{x}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$dx = 2\sqrt{x}\,du$$

$$= \int \frac{e^{1-u}}{\cancel{u}}\cdot 2\cancel{u}\,du = 2\int e^{1-u}\,du$$

$$= 2\left(\frac{1}{-1}\cdot e^{1-u}\right) + C = \underline{\underline{-2e^{1-\sqrt{x}} + C}}$$

Alt.

$$\boxed{\begin{aligned} v &= 1-u \\ dv &= -1\cdot du \end{aligned}}$$

$$= 2\int e^{v}\,\frac{1}{-1}\,dv$$

$$= -2\int e^{v}\,dv = -2e^{v} + C$$

$$= -2e^{1-u} + C = \underline{\underline{-2e^{1-\sqrt{x}} + C}}$$

Ex:

$$\int e^{\sqrt{x}}\,dx = \int e^{u}\cdot 2\underline{\sqrt{x}}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$dx = 2\sqrt{x}\,du$$

$$= \int e^{u}\cdot 2u\,du$$

$$= \int \underline{2u}\cdot e^{u}\,du$$

product → integration by parts

$$= 2ue^{u} - 2e^{u} + C$$

$$= \underline{\underline{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}} = \underline{\underline{2(\sqrt{x}-1)e^{\sqrt{x}} + C}}$$

(2) Integration by parts = "product rule" for integration

Product rule for derivation: $(u \cdot v)' = u'v + uv'$

$\Downarrow$

$$u \cdot v = \int u'v\,dx + \int uv'\,dx$$

$$\int u'v\,dx = u \cdot v - \int uv'\,dx$$

Integration by parts

$$\boxed{\int u' \cdot v\,dx = uv - \int uv'\,dx}$$

Ex: $\int x \cdot \ln x\,dx = \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$

$u = \frac{1}{2}x^2$	$v = \ln x$
$u' = x$	$v' = 1/x$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$$

Ex: $\int 2x\,e^x\,dx = e^x \cdot 2x - \int e^x \cdot 2\,dx$

$u = e^x$	$v = 2x$
$u' = e^x$	$v' = 2$

$$= 2xe^x - 2\int e^x\,dx = \underline{\underline{2xe^x - 2e^x + C}}$$

Alt: $\cancel{\int 2x\,e^x\,dx = x^2 e^x - \int x^2 \cdot e^x\,dx}$

$\cancel{u = x^2}$	$\cancel{v = e^x}$
$\cancel{u' = 2x}$	$\cancel{v' = e^x}$

Ex: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx$

$$\boxed{\int u'v \, dx = u \cdot v - \int uv' \, dx}$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx$$

$$= \underline{\underline{x \ln x - x + C}}$$

Integration formula: $\boxed{\int \ln x \, dx = x \ln x - x + C}$

Ex: $\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$

$$\boxed{\begin{array}{ll} u = e^x & v = x^2 \\ u' = e^x & v' = 2x \end{array}}$$

$$= x^2 e^x - \underline{\int 2x e^x \, dx} =$$

$$= x^2 e^x - \left(2x e^x - 2e^x\right) + C$$

$$= \underline{\underline{x^2 e^x - 2x e^x + 2e^x + C}}$$

(3) Integration of rational functions / fractions

Ex: i) $\int \frac{2}{1-x}\,dx$ ii) $\int \frac{2x}{1-x^2}\,dx$ iii) $\int \frac{2}{1-x^2}\,dx$

i) $\int \frac{2}{1-x}\,dx = \int \frac{2}{u}\frac{1}{(-1)}\,du = \frac{2}{-1}\int \frac{1}{u}\,du$

$\boxed{\begin{array}{l} u = 1-x \\ du = -1\cdot dx \end{array}}$

$dx = \frac{1}{(-1)}\cdot du$

$= -2\ln|u| + C$

$= -2\ln|1-x| + C$

$a \neq 0$: $\int \frac{A}{ax+b}\,dx = \int \frac{A}{u}\cdot\frac{1}{a}\,du = \frac{A}{a}\int\frac{1}{u}\,du$

$\boxed{\begin{array}{l} u = ax+b \\ du = a\cdot dx \end{array}}$

$dx = \frac{1}{a}\cdot du$

$= \frac{A}{a}\ln|u| + C = \frac{A}{a}\ln|ax+b| + C$

$$\boxed{\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C \qquad (a \neq 0)}$$

Ex: $\int \frac{x}{1-x}\,dx = \int -1 + \frac{1}{1-x}\,dx = -x + \frac{1}{-1}\cdot\ln|1-x| + C$

poly. division

if degree numerator $\geq$ degree denominator

$= -x - \ln|1-x| + C$

$$\begin{array}{l} x : (-x+1) = -1 \\ -(x-1) \\ \hline \quad 1 \end{array}$$

$\Rightarrow \frac{x}{1-x} = -1 + \frac{1}{1-x}$

ii) $\displaystyle\int \frac{2x}{1-x^2}\,dx = \int \frac{\cancel{2x}}{u}\cdot\frac{1}{\cancel{-2x}}\,du = -\int \frac{1}{u}\,du$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}}$$

$dx = \frac{1}{-2x}\,du$

$= -\ln|u| + C$

$= -\ln|1-x^2| + C$

iii) $\displaystyle\int \frac{2}{1-x^2}\,dx$ ~~$= \int \frac{2x}{u}\cdot\frac{1}{-2x}\,du = -\int \frac{1}{xu}\,du$~~

~~$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}}$ $\to x^2 = 1-u$, $x = \pm\sqrt{1-u}$~~

not possible with substitution

Method: Partial fractions

Ex: $\displaystyle\frac{2}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x} \quad \Big|\cdot(1-x^2)$ A, B constants, unknown.

$1-x^2 = (1+x)(1-x)$

$2 = \frac{A}{\cancel{1+x}}\cdot\cancel{(1+x)}(1-x) + \frac{B}{\cancel{1-x}}\cdot(1+x)\cancel{(1-x)}$

$2 = A\cdot(1-x) + B(1+x)$

Alt I: $2 = A - Ax + B + Bx$

$0\cdot x + 2 = (A+B) + (-A+B)x$ $\downarrow$ comparing coefficients

$0 = -A+B \Rightarrow A = B$

$2 = A+B$

$\frac{2}{2} = \frac{A+A = 2A}{2}$

$A = 1$
$B = 1$

Conclusion:

$$\boxed{\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x}}$$

iii) $\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1+x} + \frac{1}{1-x}\,dx$

$= \int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx$

$= \frac{1}{1} \ln|1+x| + \frac{1}{-1} \ln|1-x| + C$

$= \underline{\underline{\ln|1+x| - \ln|1-x| + C}} = \ln \frac{|1+x|}{|1-x|} + C$

Alt 2: $2 = A(1-x) + B(1+x)$

$x=1$: $2 = A \cdot 0 + B \cdot 2$ $\quad B=1$

$x=-1$: $2 = A \cdot (+2) + B \cdot 0$ $\quad A=1$

linear expression on both sides → straight lines → straight line is determined by two points

($x=1$, $x=-1$ in this case)