Plan

1 Definite integrals and antiderivatives
2 Applications of the definite integral
3 Improper integrals

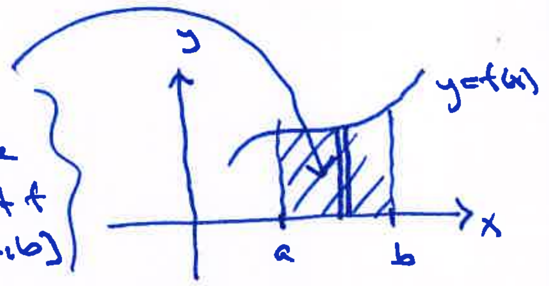
① Definite integrals

Assume:

i) f is cont. fn. on $[a,b]$

ii) $f(x) \geq 0$ for all x in $[a,b]$

$$\int_a^b f(x)\,dx = \left\{ \begin{array}{l} \text{area} \\ \text{under the} \\ \text{graph of } f \\ \text{in } [a,b] \end{array} \right\}$$

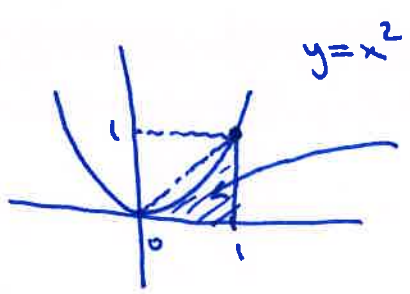


can be computed (approximated) with Riemann sums

New defn: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ when $F'(x) = f(x)$

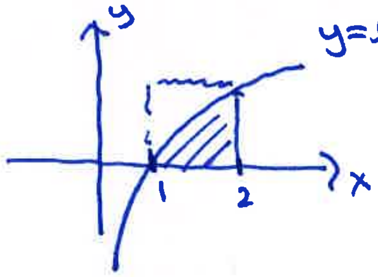
Ex: $$\underbrace{\int_0^1 x^2\,dx}_{\text{definite integral}} = \Big[\underbrace{\tfrac{1}{3}x^3 + C}_{F(x)}\Big]_0^1 = \underbrace{\left(\tfrac{1}{3}\cdot 1^3 + C\right)}_{F(1)} - \underbrace{\left(\tfrac{1}{3}\cdot 0^3 + C\right)}_{F(0)}$$

$$= \tfrac{1}{3} + \cancel{C} - 0 - \cancel{C} = \underline{\underline{1/3}}$$



$$\text{Area} = \int_0^1 x^2\,dx = \underline{\underline{1/3}}$$

$$\int_1^2 \ln x\,dx = \Big[\underbrace{x\ln x - x}_{F(x)}\Big]_1^2 = \underbrace{(2\ln 2 - 2)}_{F(2)} - \underbrace{(1\cdot\ln 1 - 1)}_{F(1)}$$



$$= 2\ln 2 - 2 - 0 + 1$$
$$= 2\ln 2 - 1 \simeq 0.386$$

$$\text{Area} \simeq \frac{1\cdot\ln 2}{2} \simeq 0.35$$

$$\int \ln x\,dx = \int \underbrace{1}_{u'}\cdot\underbrace{\ln x}_{v}\,dx = \underbrace{x}_{u}\cdot\underbrace{\ln x}_{v} - \int \underbrace{x}_{u}\cdot\underbrace{\tfrac{1}{x}}_{v'}\,dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x\ln x - \int 1\,dx = x\ln x - x + C$$

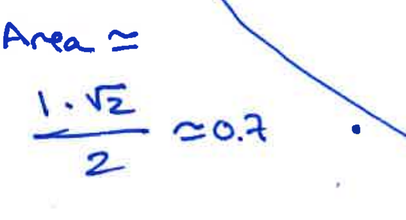
Ex:

$$\int_{x=0}^{x=1} x\sqrt{x^2+1}\,dx = \int \cancel{x}\sqrt{u}\,\frac{1}{2\cancel{x}}du = \int_{u=1}^{u=2} \frac{1}{2}u^{1/2}\,du$$

$$\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$$

$x=0$: $u=1$

$x=1$: $u=2$



$$= \left[\frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\right]_1^2$$

$$\left[\frac{(x^2+1)\sqrt{x^2+1}}{3}\right]_0^1 \longrightarrow = \left[\frac{u\sqrt{u}}{3}\right]_1^2$$

Area $\simeq$

$$\frac{1\cdot\sqrt{2}}{2} \simeq 0.7$$

$$= \frac{2\sqrt{2}}{3} - \frac{1\cdot\sqrt{1}}{3}$$

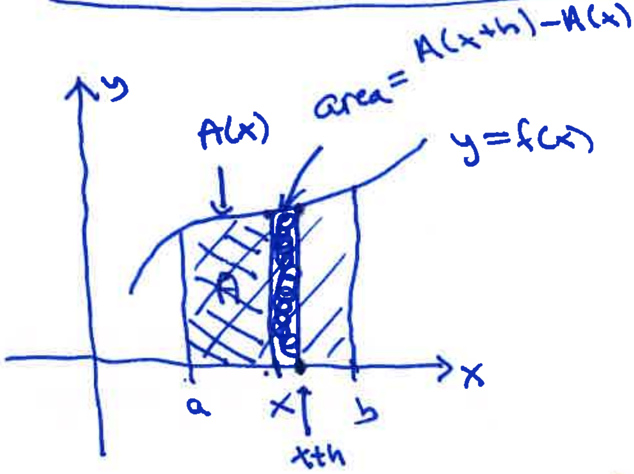
$$\frac{2\sqrt{2}}{3} - \frac{1}{3} \longrightarrow = \frac{2}{3}\sqrt{2} - \frac{1}{3} = \frac{1}{3}\left(2\sqrt{2}-1\right)$$

$$\simeq 0.609$$

Theorem: If f is a continuous fn. on $[a,b]$ such that $f(x) \geq 0$ for x in $[a,b]$, then

$$\left\{ \begin{array}{l} \text{area under the graph} \\ \text{of } f(x) \text{ in the} \\ \text{interval } [a,b] \end{array} \right\} = \int_a^b f(x)\,dx = F(b) - F(a)$$

when $F'(x) = f(x)$ is a anti-derivative.



$$A = \left\{ \begin{array}{l} \text{area under } y=f(x) \\ \text{in } [a,b] \end{array} \right\}$$

$$\int_a^b f(x)\,dx = \Big[\; A(x) \; \Big]_a^b$$

$$= A(b) - A(a)$$

$$= A - 0 = A$$

Explanation:

Define an area function:

$$A(x) = \left\{ \begin{array}{l} \text{area under } y=f(x) \\ \text{in } [a,x] \end{array} \right\}$$

Facts:

$A(a) = 0$ $\qquad A(b) = A$

$$A'(x) \simeq \frac{A(x+h) - A(x)}{h} \simeq \frac{\text{area of thin strip}}{h} \simeq \frac{h \cdot f(x)}{h} = f(x)$$

Note: $A'(x) = f(x)$

$A(x)$ is antiderivative of $f(x)$

Eivind Eriksen, Office B4-032, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/EBA2910/

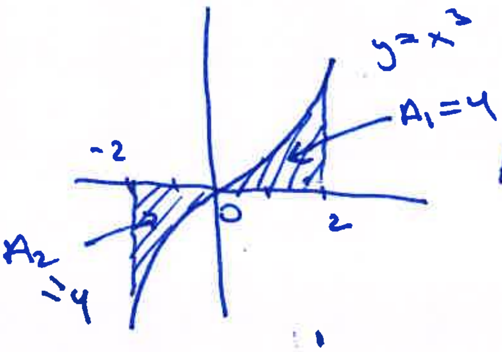
② Applications of the definite integral

- Computation of areas between graphs
- Economic applications

Ex:

$$\int_{-2}^{2} x^3\,dx = \left[x^4/4\right]_{-2}^{2} = \frac{2^4}{4} - \frac{(-2)^4}{4} = -4 + 4 = 0$$

$$= \frac{16}{4} - \frac{16}{4} = \underline{\underline{0}} = -A_2 + A_1$$



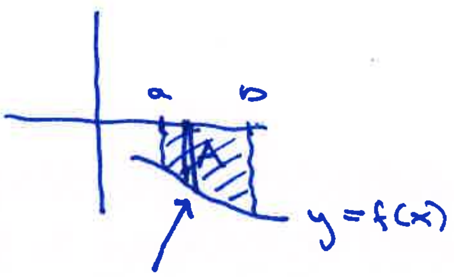
$$A_1 = \int_0^2 x^3\,dx = \left[x^4/4\right]_0^2 = \frac{2^4}{4} - \frac{0^4}{4}$$

$$= \frac{16}{4} - 0 = \underline{\underline{4}}$$

$$\int_{-2}^{0} x^3\,dx = \left[x^4/4\right]_{-2}^{0} = \frac{0^4}{4} - \frac{(-2)^4}{4}$$

$$= 0 - 4 = \underline{-4}$$

When $f(x) \le 0$ in $[a,b]$:



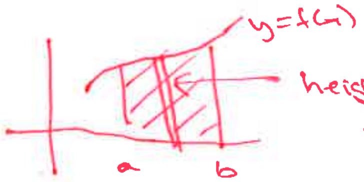
height of this rectangle: $0 - f(x)$

$= -f(x)$

Area between the x-axis and the graph of $y=f(x)$ in $[a,b]$:

$$\boxed{\begin{array}{c} A = \int_a^b -f(x)\,dx \\ \Updownarrow \\ \int_a^b f(x)\,dx = -A \end{array}}$$

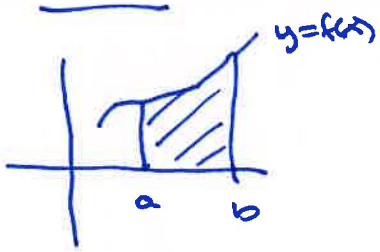
$f(x) \ge 0$:



height $=$ $f(x) - 0$

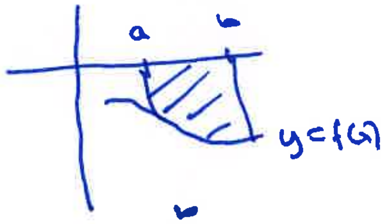
$= f(x)$

i) $f(x) \geq 0$



$$A = \int_a^b f(x)\,dx$$

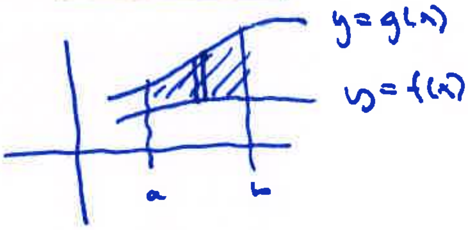
ii) $f(x) \leq 0$



$$A = \int_a^b -f(x)\,dx$$

$$-A = \int_a^b f(x)\,dx$$

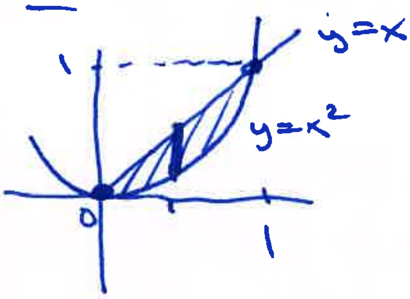
iii) $f(x) \leq g(x)$



$$A = \int_a^b g(x) - f(x)\,dx$$

$$= \int_a^b [g(x) - f(x)]\,dx$$

Ex: Area between $y=x$ and $y=x^2$ in $[0,1]$



$$A = \int_0^1 x - x^2\,dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1$$

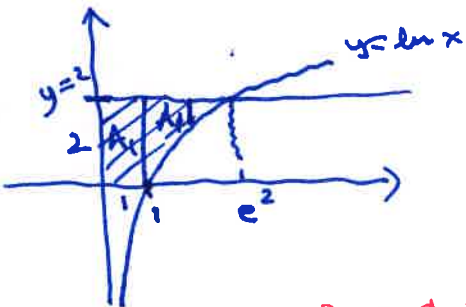
$$= \left(\tfrac{1}{2}\cdot 1^2 - \tfrac{1}{3}\cdot 1^3\right) - 0 = \tfrac{1}{2} - \tfrac{1}{3}$$

$$= \tfrac{3}{6} - \tfrac{2}{6} = \underline{\underline{\tfrac{1}{6}}}$$

$$\simeq 0.167$$

$0 < x < 1$: $x > x^2$

Ex: Area bounded by $y = \ln x$, $y=2$, the y-axis and the x-axis:



$$A = A_1 + A_2 = 1\cdot 2 + A_2$$

$$= 2 + \int_1^{e^2} 2 - \ln x\,dx$$

$$= 2 + \left[2x - (x\ln x - x)\right]_1^{e^2}$$

$y=2$ and $y=\ln x$:

$$= 2 + \left[3x - x\ln x\right]_1^{e^2}$$

$2 = \ln x$

$$= 2 + (3e^2 - e^2\cdot 2) - (3 - \cancel{1\cdot 0})$$

$e^2 = x$

$x = e^2$

$$= 2 + e^2 - 3 = \underline{\underline{e^2 - 1}} \simeq 6.389$$

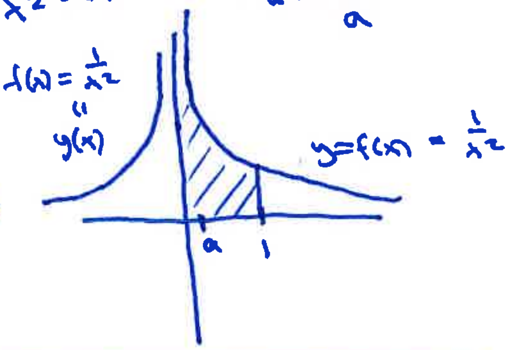
Eivind Eriksen, Office B4-032, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/EBA2910/

When $f(x)$ is not continuous on $[a,b]$

ii) $a=-\infty$ or $b=\infty$

3) Improper integrals

Ex: $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0}\int_a^1 \frac{1}{x^2}\,dx$



Note: $f(x)=1/x^2$ not defn. at $x=0$

$$\int_a^1 \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_a^1 = \left[-\frac{1}{x}\right]_a^1 = \left(-\frac{1}{1}\right)-\left(-\frac{1}{a}\right)$$

$\frac{1}{x^2} = x^{-2}$

$$= -1+\frac{1}{a}$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0}\int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0}\left(-1+1/a\right)$$

$$= \underline{\underline{\infty}}$$

Ex: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left(1-\frac{1}{a}\right) = \underline{\underline{1}}$

Note: $[1,\infty)$ is not bounded

$$\int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a = \left(-\frac{1}{a}\right)-\left(-\frac{1}{1}\right) = -\frac{1}{a}+1 = 1-\frac{1}{a}$$