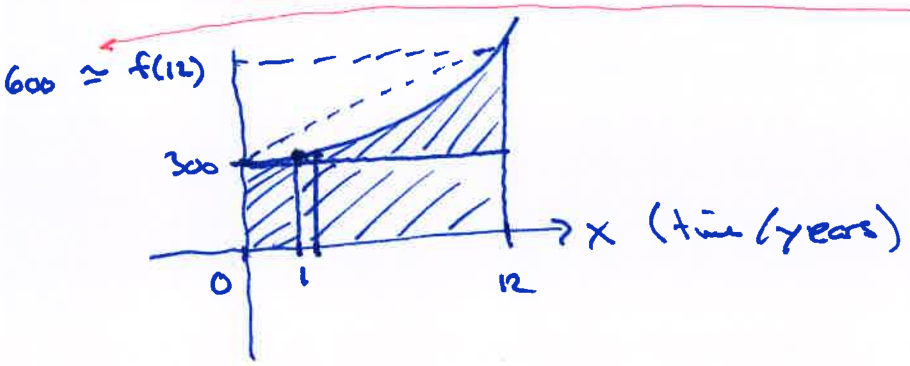
Plan

1 Economic applications of the definite integral
2 Examples and problems

① Economic applications

(a) Continuous cash flows

Ex: $f(x) = 300 \cdot 1.06^x$ (cash flow, MNOK/year)



Rule of 72: Doubling takes approx. $\frac{72}{6} = 12$ years when something grows with 6% per year

Total cash flow in 12 years: = area under the graph in [0,12]

$$= \int_0^{12} f(x)\,dx = \int_0^{12} 300 \cdot \underline{1.06^x}\,dx$$

$$= \left[300 \cdot \frac{1}{\ln(1.06)} 1.06^x\right]_0^{12}$$

$$= \frac{300}{\ln(1.06)}\left[1.06^x\right]_0^{12} = \frac{300}{\ln(1.06)} \cdot \left(1.06^{12} - 1\right)$$

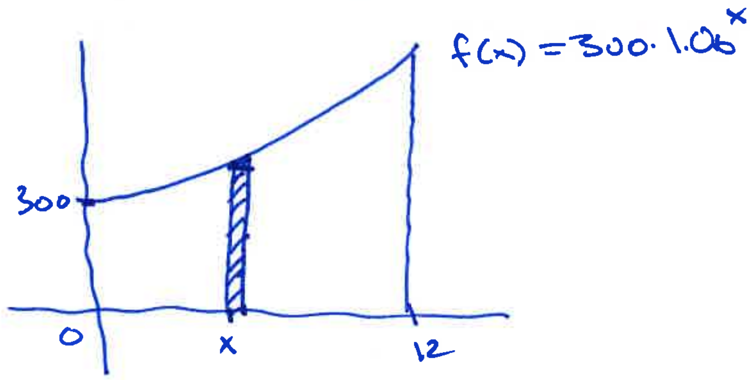
$$\simeq \underline{5.211} \text{ MNOK}$$

Remember:

$\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)} \cdot a^x + C$

NPV of a continuous cash flow:



$\leftarrow$ r = discount rate = 10%
continuous discounting

NPV: $\displaystyle\int_0^{12} f(x)\, e^{-rx}\, dx = \int_0^{12} \underbrace{300 \cdot 1.06^x}_{f(x)} \cdot e^{-0.10x}\, dx$

$\displaystyle = 300 \int_0^{12} 1.06^x \cdot e^{-0.10x}\, dx$

$\displaystyle = 300 \cdot \int_0^{12} e^{\ln(1.06)x} \cdot e^{-0.10x}\, dx$

$\displaystyle = 300 \int_0^{12} e^{(\ln(1.06)-0.10)x}\, dx$

$\displaystyle = 300 \left[\frac{1}{\ln 1.06 - 0.1} \cdot e^{(\ln 1.06 - 0.1)x}\right]_0^{12}$

$\displaystyle = \frac{300}{\ln(1.06)-0.10} \cdot \left(e^{(\ln 1.06 - 0.1)\cdot 12} - 1\right)$

$\displaystyle = \underbrace{\frac{300}{\ln(1.06)-0.10}}_{-7.189} \underbrace{\left(1.06^{12}/e^{1.2} - 1\right)}_{\simeq -0.394}$

$\simeq \underline{\underline{2.832}}$ MNOK

$$1.06^x = \left(e^{\ln(1.06)}\right)^x = e^{\ln(1.06)x}$$

$u = (\ln 1.06 - 0.10)x$

$du = (\ln 1.06 - 0.10) \cdot dx$

$\displaystyle\int e^{(1.06 - 0.1)x}\, dx$

$\displaystyle = \int e^u \frac{du}{\ln 1.06 - 0.10}$

$\displaystyle = \frac{1}{\ln 1.06 - 0.1} e^u + C$

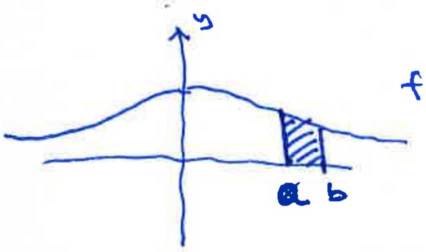
Formulas: Total cash flow : $\int_0^T f(x)\,dx$

NPV of cash flow: $\int_0^T f(x)\, e^{-rx}\,dx$

f(x): cash flow per time unit

r: discount rate

(b) Probabilities (continuous stochastic variables)



$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$ standard normal probability distribution

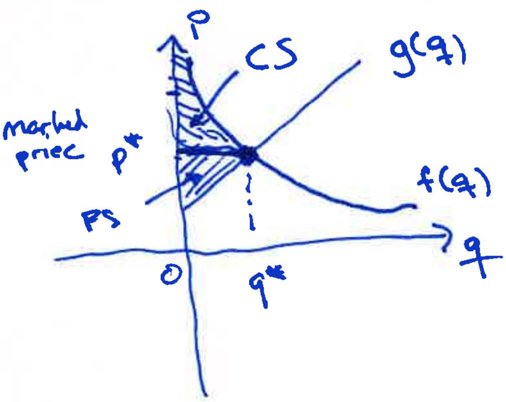
$X \sim N(0,1)$

$$p(a \le X \le b) = \int_a^b f(x)\,dx$$

(c) Consumer / Producer surplus

$p = f(q)$ demand function (inverse)

$p = g(q)$ supply function (inverse)



CS (consumer surplus)

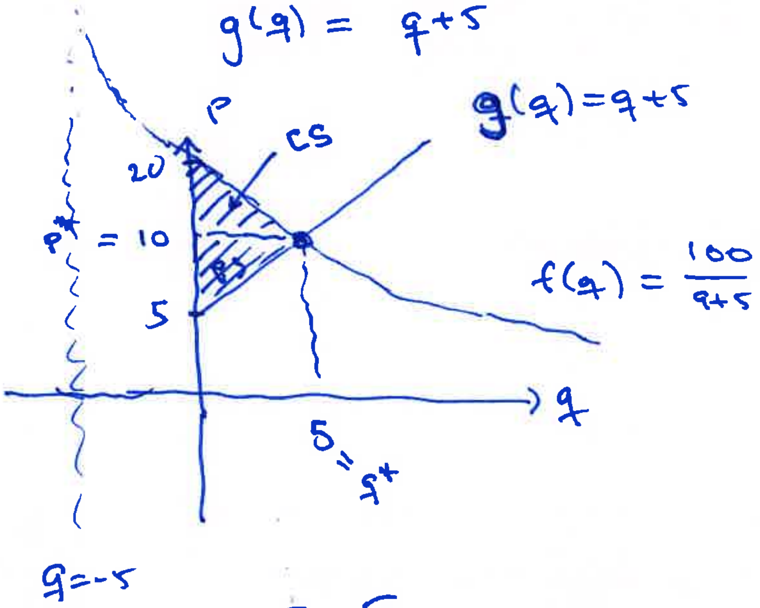
$$CS = \int_0^{q^*} f(q) - p^*\,dq$$

PS (producer surplus)

$$PS = \int_0^{q^*} p^* - g(q)\,dq$$

Ex: $f(q) = \dfrac{100}{q+5}$

$g(q) = q+5$



<u>Market price:</u>

$$f(q) = g(q)$$

$$\frac{100}{q+5} = q+5$$

$$100 = (q+5)^2$$

$$q+5 = \pm\sqrt{100} = 10$$

$$q = 10-5 = 5$$

$$\underline{q^* = 5}$$

$$PS = \frac{5 \cdot 5}{2} = \underline{\underline{12.5}} = \int_0^5 10 - (q+5)\, dq$$

$$= \int_0^5 5 - q \, dq = \left[5q - \tfrac{1}{2} q^2\right]_0^5$$

$$= \left(5 \cdot 5 - \tfrac{1}{2} \cdot 5^2\right) - 0$$

$$= 25 - \tfrac{1}{2} \cdot 25 = \underline{\underline{12.5}}$$

$$CS = \int_0^5 \frac{100}{q+5} - 10 \, dq = \left[100 \ln(q+5) - 10q\right]_0^5$$

$$= \left(100 \ln(10) - 50\right) - \left(100 \cdot \ln(5) - 0\right)$$

$$= 100 \ln(10) - 50 - 100 \ln(5)$$

$$= 100 \ln\left(\tfrac{10}{5}\right) - 50 = \underline{\underline{100 \ln 2 - 50}}$$

$$\simeq \underline{\underline{19}}$$

$$\boxed{\ln a - \ln b = \ln a/b}$$

② Examples:

$$\int 30\, \underset{x^1}{x}\, \underset{x^{0.5}}{\sqrt{x}}\, dx = \int 30\, x^{3/2}\, dx = 30 \cdot \frac{x^{5/2}}{5/2} + C$$

$$= 30 \cdot \frac{2}{5}\, x^2 \cdot x^{0.5} + C = 12x^2\sqrt{x} + C$$

$$\int \underset{v}{x}\, \underset{u'}{e^{-x}}\, dx = -e^{-x}\cdot x - \int -e^{-x}\cdot 1\, dx$$

$u = -e^{-x}$ $\quad v = x$

$u' = e^{-x}$ $\quad v' = 1$

(integration by parts)

$$\int u'v\, dx = uv - \int uv'\, dx$$

$$= -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + C$$

$$\int \frac{6-3x}{4-9x^2}\, dx = \int \frac{2}{2+3x}\, dx + \int \frac{1}{2-3x}\, dx = \frac{2}{3}\ln(2+3x) + \frac{1}{-3}\ln(2-3x) + C$$

$$4-9x^2 = (2+3x)(2-3x)$$

$$a^2-b^2 = (a+b)(a-b)$$

$4-9x^2 = 0$

$\frac{-9x^2}{-9} = \frac{-4}{-9}$

$x^2 = 4/9$

$x = \pm\sqrt{4/9} = \pm 2/3$

$4-9x^2 = -9(x-2/3)(x+2/3)$

Partial fractions:

$$\frac{6-3x}{(2+3x)(2-3x)} = \frac{A^{=2}}{2+3x} + \frac{B^{=1}}{2-3x}$$

$$6-3x = A(2-3x) + B(2+3x)$$

$$6-3x = \underbrace{(2A+2B)}_{=6} + \underbrace{(-3A+3B)}_{=-3}x$$

$2A+2B = 6 \quad :2$

$-3A+3B = -3 \quad :3$

$A+B = 3$

$-A+B = -1$

$2B = 2$

$B = 1$

$A = 2$