Plan

1 Systems of equations
2 Linear systems and Gaussian elimination

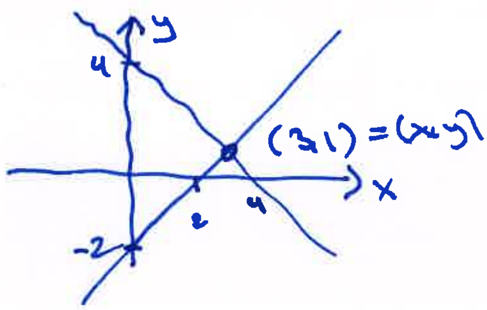
1 Systems of equations

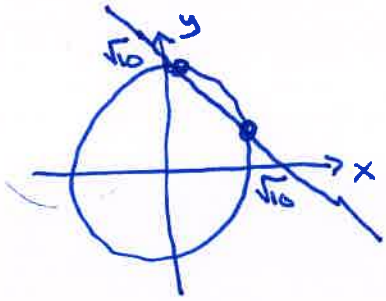
Linear

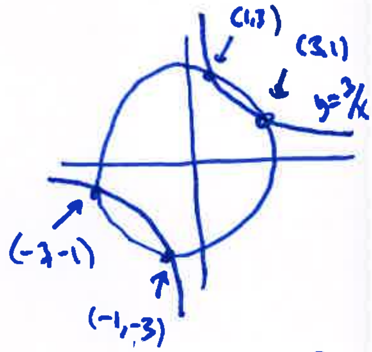
Ex: i) $x+y=4$ 1 ; $x-y=2$ 1

ii) $x^2+y^2=10$ 2 ; $x+y=4$ 1

iii) $x^2+y^2=10$ 2 ; $xy=3$ 2







$x+y=4 \Rightarrow y=-x+4$

$x-y=2 \Rightarrow y=x-2$

Solution methods:

Elimination:

$x+y=4$

$+\ x-y=2$

$2x \quad =6$

$x=3$

$y=1$

$(x,y)=(3,1)$

Substitution:

$x+y=4 \Rightarrow y=4-x$

$x-y=2$

$x-(4-x)=2$

$2x-4=2$

$2x=6$

$x=3$

$y=1$

$(x,y)=(3,1)$

ii)

$x+y=4 \Rightarrow y=4-x$

$x^2+(4-x)^2=10$

$x^2+16-8x+x^2=10$

$2x^2-8x+6=0$

$x^2-4x+3=0$

$x=3, \quad x=1$

$x=\dfrac{4\pm\sqrt{(-4)^2-4\cdot1\cdot3}}{2\cdot1}$

$=\dfrac{4\pm2}{2}=3,1$

$x=3 \quad x=1$

$y=1 \quad y=3$

$(x,y)=(3,1),(1,3)$

$xy=3 \Rightarrow y=3/x$

iii) $y=3/x$

$x^2+(3/x)^2=10$

$x^2+\dfrac{9}{x^2}=10$

$x^4+9=10x^2$

$x^4-10x^2+9=0$

$(x^2)^2-10(x^2)+9=0$

$x^2=9, \ x^2=1$

$x=\pm3, \ x=\pm1$

$(x,y)=(3,1),(-3,-1),(1,3),(-1,-3)$

Defn: An <u>$m \times n$ linear system</u> in the variables $x_1, x_2, \dots, x_n$ is a system of m linear equations in $x_1, x_2, \dots, x_n$. It has the form

$$m \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right.$$

$\underbrace{\qquad\qquad\qquad\qquad}_{n \text{ variables}}$

where $a_{11}, a_{12}, \dots, a_{mn}$ and $b_1, b_2, \dots, b_m$ are given numbers.

Ex:

$$\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 7 \\ x_1 - x_2 \qquad + 2x_4 = 10 \\ x_1 + x_2 - x_3 \qquad = 3 \end{array}$$

3×4 lin. system

Ex:

$$\begin{array}{l} x + y + z = 3 \\ x + 2y + 4z = 7 \\ x + 3y + 9z = 13 \end{array}$$

3×3 linear system

Methods for solving linear systems:

$$\begin{array}{l} x + y + z = 3 \\ x + 2y + 4z = 7 \\ x + 3y + 9z = 13 \end{array}$$

① $x = \underline{3 - y - z}$

② $(3 - y - \cancel{z}) + 2y + 4z = 7$

$\boxed{y + 3z = 4}$

③ $(3 - y - z) + 3y + 9z = 13$

$\boxed{2y + 8z = 10}$

$$\begin{array}{l} y + 3z = 4 \\ 2y + 8z = 10 \end{array}$$

$y = \underline{4 - 3z}$

$2(4 - 3z) + 8z = 10$

$2z = 2$

$\underline{z = 1}$

$\underline{y = 1}$

$\underline{x = 1}$

<u>One solution:</u>

$(x, y, z) = \underline{(1, 1, 1)}$

② Gaussian elimination: Method for solving any linear system

Method:

① Write down the augmented matrix of the system

② Use elementary row operations until you have an echelon form

③ Use back substitution to solve the system

Elementary row operations:

i) Switch two rows

ii) Multiply a row with $c \neq 0$

iii) Add a multiple of one row to another row

Ex:

$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+3y+9z &= 13 \end{aligned}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \; -1$$

$(-1 \;\; -1 \;\; -1 \mid -3)$

augmented matrix

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right) \; -1$$

$(-1 \;\; -1 \;\; -1 \mid -3)$

$\downarrow$

$(0 \;\; -2 \;\; -6 \mid -8)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \; -2$$

$$\begin{aligned} y+3z &= 4 \\ 2y+8z &= 10 \end{aligned}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

Echelon form:

Defn: the first non-zero element in a row is called a pivot

Echelon form: ① all entries below a pivot are zero

② zero row in the

Back substitution:

$$\begin{aligned} x + y + z &= 3 \\ y + 3z &= 4 \\ 2z &= 2 \end{aligned}$$

- start with last equation
- work backwards, and substitute the variables we have solved for

$2z = 2 \Rightarrow z = 1$

$y + 3 \cdot 1 = 4 \Rightarrow y = 1$

$x + 1 + 1 = 3 \Rightarrow x = 1$

One solution:

$(x, y, z) = (1, 1, 1)$

How to determine the number of solutions

Ex:

$$\begin{aligned} x + y + z &= 4 \\ x - y + z &= 2 \\ x + 5y + z &= 8 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \\ 1 & 5 & 1 & 8 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 4 & 0 & 4 \end{array}\right) \begin{matrix} \\ 2 \\ \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

$$\begin{aligned} x + y + z &= 4 \\ -2y &= -2 \end{aligned}$$

$-2y = -2 \Rightarrow y = 1$

$x + 1 + z = 4$

$\Rightarrow x = 3 - z$

pivots $\rightarrow$ basic variables: x, y

var. cols. without pivots $\rightarrow$ free variables: z

Concl: $(x, y, z) = (3 - z, 1, z)$

where z is free

infinitely many solutions

Ex:
$$x+y+z=4$$
$$x-y+z=2$$
$$x+5y+z=9$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \\ 1 & 5 & 1 & 9 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$x+y+z=4$$
$$-2y=-2$$
$$0\cdot x+0\cdot y+0\cdot z=1$$

no solution

In general:

pivot in the last column of an echelon form

$\Updownarrow$

no solutions