Plan

1. Linear systems revisited: Number of solutions
2. Computing with matrices and vectors
3. Determinants

① Linear systems: Number of solutions

Ex:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\ x_1 - 2x_2 - x_3 \quad + 4x_5 &= 8 \\ 2x_1 + x_2 - 5x_3 + 7x_4 \quad &= 11 \end{aligned}$$

Gaussian elimination:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 1 & -2 & -1 & 0 & 4 & 8 \\ 2 & 1 & -5 & 7 & 0 & 11 \end{array}\right) \begin{array}{c} -1 \quad -2 \end{array}$$

$\downarrow$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -3 & -2 & -1 & 3 & -9 \\ 0 & -1 & -7 & 5 & -2 & -23 \end{array}\right)$$

$\downarrow$

$$-3\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & -3 & -2 & -1 & 3 & -9 \end{array}\right)$$

$\downarrow$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ \cancel{0} & \cancel{0} & \cancel{-5} & \cancel{-4} & \cancel{0} & \cancel{-60} \\ 0 & 0 & 19 & -16 & 9 & 60 \end{array}\right)$$

echelon form

pivot positions

(1,1), (2,2), (3,3)

two degrees of freedom (x_4, x_5 free)

infinitely many solutions

Defn: A pivot position = a position where there is a pivot in the echelon form.

Result: For any linear system, the pivot position determine the number of solutions:

i) pivot position in the last column: no solutions

ii) no pivot position in the last column: there are solutions

(a) pivot position in all variable columns: one solution

(b) there are variable columns without pivot positions: infinitely many solutions

Theorem: Any linear system has either

i) no solutions } inconsistent

ii) one unique solution

iii) infinitely many solutions } consistent

(2) Computing with vectors and matrices

Defn: An $m \times n$ - matrix is a rectangular array of numbers with m rows and n columns.

Ex:

$$A = \overbrace{\begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \end{pmatrix}}^{3} \Big\} 2$$

2×3 - matrix

row 1, col. 2

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

2×3 - matrix

Addition: $A + B$

Subtraction: $A - B$

} defined if A, B have the same size

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \end{pmatrix}$$

(position by position)

Scalar multiplication: $r \cdot A$

r scalar
A matrix

r = scalar (number)

A = matrix

Ex:

$$2 \cdot \begin{pmatrix} 1 & 4 \\ -1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ -2 & 4 \\ 0 & 2 \end{pmatrix}$$

Defn: An n-vector is a matrix with n rows and 1 column (column vector).

Write vectors as: $\underline{v}$ = boldface v $= \vec{v}$

Ex: $\underline{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ 3-vector viewed as a column vector

Vector operations:

- addition: $\underline{v} + \underline{w}$
- subtraction: $\underline{v} - \underline{w}$
- scalar multiplication: $r \cdot \underline{v}$

Ex:

$$\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

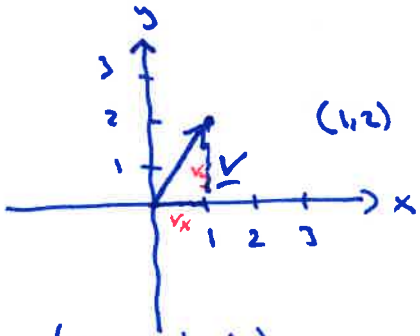
$$\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$$

$$2 \cdot \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

$$-1 \cdot \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

Geometric interpretation of vectors

Ex: $\underline{v} = \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}v_x\\v_y\end{pmatrix}$ corresponds to an arrow from $(0,0)$ to (v_x, v_y)



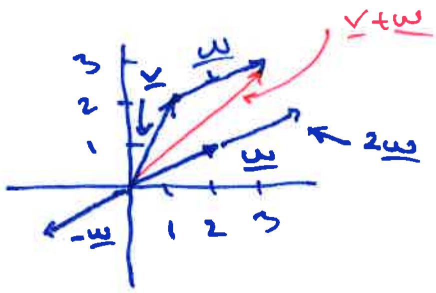
A vector is quantity with an length (magnitude) and a direction

Length: $\|\underline{v}\| = \sqrt{v_x^2 + v_y^2}$ $\qquad \left\|\begin{pmatrix}1\\2\end{pmatrix}\right\| = \sqrt{1^2+2^2} = \underline{\underline{\sqrt{5}}}$

In general: $\underline{v} = \begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}$: $\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

Ex: $\underline{v} = \begin{pmatrix}1\\2\end{pmatrix}$ $\underline{w} = \begin{pmatrix}2\\1\end{pmatrix}$:

$\underline{v}+\underline{w} = \begin{pmatrix}3\\3\end{pmatrix}$ $\qquad 2\underline{w} = \begin{pmatrix}4\\2\end{pmatrix}$ $\qquad -\underline{w} = \begin{pmatrix}-2\\-1\end{pmatrix}$, $-\underline{w} = -1 \cdot \underline{w}$



③ Determinants

$A \rightsquigarrow \det(A) = |A|$

$n \times n$ matrix (square) — a number

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (2×2) $\quad \det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2\cdot 2 - 1\cdot 1 = \underline{\underline{3}}$

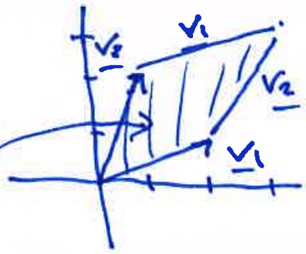
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\boxed{\underline{\text{Case } n=2:} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

Interpretation:

Ex: $A = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$

$\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



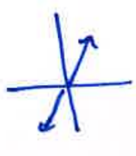
$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \pm$ area of the parallellogram spanned by $\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

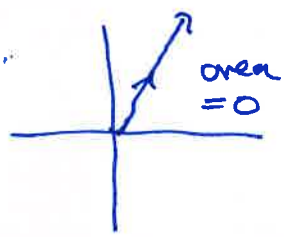
$\| \quad 2\cdot 2 - 1\cdot 1 \quad \| \quad 3 \quad \| \quad$ area of

Ex: $\begin{array}{cc} \underline{v_2} & \underline{v_1} \end{array}$ $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1\cdot 1 - 2\cdot 2 = \underline{\underline{-3}}$

Ex: $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 2 = \underline{\underline{0}}$

$\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = \underline{\underline{0}}$





area $= 0$

Result: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ satisfy: one of the vectors is a scalar multiple of the other

The general case: A $n \times n$-matrix

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \begin{matrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$$

$= 1 \cdot 2 \cdot 9 + 1 \cdot 4 \cdot 1 + 1 \cdot 1 \cdot 3$
$- 1 \cdot 2 \cdot 1 - 3 \cdot 4 \cdot 1 - 9 \cdot 1 \cdot 1$
$= 18 + 4 + 3 - 2 - 12 - 9 = \underline{\underline{2}}$

in general: sum/difference of $n!$ terms of degree n

$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

If $n > 3$, this method does not work!

Method: Cofactor expansion (general method)

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$

$= 1 \cdot (+1) \cdot M_{11} + 1 \cdot (-1) \cdot M_{12} + 1 \cdot (+1) \cdot M_{13}$

$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= +(2 \cdot 9 - 3 \cdot 4) - (1 \cdot 9 - 1 \cdot 4) + (1 \cdot 3 - 1 \cdot 2)$

$= +6 - 5 + 1 = \underline{\underline{2}}$

cofactor expansion along the first row

Cofactors:

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$, where M_{ij} (minor) is the determinant of the submatrix you get when you delete row i, col. j.

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

- Cofactor expansion along any row or column give the same result.
- Any determinant can be computed using cofactor expansion.

Ex:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 2 & 4 & 8 \\ 3 & 9 & 27 \\ -1 & 1 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 & 8 \\ 1 & 9 & 27 \\ 1 & 1 & -1 \end{vmatrix}$$

$$+ \ \ldots \qquad - \ \ldots \, .$$