Plan

① Determinants and linear systems

Ex:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= +1\Big(-1 \cdot \big(\underbrace{1\cdot(-1) - 1\cdot 1}_{-2}\big)\Big) + 1\Big(-1\big(\underbrace{1\cdot(-1) - 1\cdot 1}_{-2}\big)\Big)$$

$$= 1 \cdot 2 + 1 \cdot 2 = \underline{\underline{4}}$$

Alternative method: Gaussian elimination

Ex:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{matrix} \\ \\ \big]{-1} \\ \\ \end{matrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{matrix} \\ \\ \\ \big]{-1} \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = E$$

① $|E| = 1 \cdot 1 \cdot (-2) \cdot (-2) = \underline{4}$

② $|A| = |E| = \underline{\underline{4}}$

Note: ① If A is an upper triangular matrix (all entries below the main diagonal are zero), then $|A|$ is the product of the diagonal entries.
All echelon forms are upper triangular

Why?

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= 1 \cdot \left(1 \cdot \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} \right) = 1 \cdot 1 \cdot (-2) \cdot (-2) = \underline{\underline{4}}$$

Ex:

$$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot 2 \cdot 0 = \underline{\underline{0}}$$

Note 2: If $A \to B$ is an elementary row operation, then:

i) switch two rows : $|B| = -|A|$

ii) multiply one row with $c \neq 0$: $|B| = c \cdot |A|$

iii) add a multiple of one row to another row : $|B| = |A|$

Linear systems: $A \cdot \underline{x} = \underline{b}$ (matrix form)

Ex:

$$\boxed{\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+3y+9z &= 13 \end{aligned}}$$

3x3 linear system

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

coeff. matrix

3x3 matrix

If we start with an nxn linear system (# equations = # variables), then A is an nxn-matrix, and we can compute $|A|$.

Note:

$|A| \neq 0$: one solution (unique)

$|A| = 0$: no solutions or infinitely many solutions

Ex:

$$\begin{aligned} x+y+z&=3\\ x+2y+4z&=7\\ x+3y+9z&=13 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9 \end{vmatrix} = 1(6)-1(5)+1(1) = \underline{2}$$

$|A| \neq 0 \Rightarrow$ one unique solution

$$\begin{pmatrix} 1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 2 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 2 \end{pmatrix}$$

Ex:

$$\begin{aligned} x+y+z&=3\\ x+2y+4z&=7\\ 2x+3y+5z&=10 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1\\ 1 & 2 & 4\\ 2 & 3 & 5 \end{vmatrix} = 1(-2)-1(-3)+1(-1) = \underline{0}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & 4 & 7\\ 2 & 3 & 5 & 10 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 0 & 1 & 3 & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 0 \end{array}\right)$$

(2) Linear systems with parameters

Ex:

$$\begin{aligned} x+y&=4\\ x+ay&=6 \end{aligned}$$

endogenous

x, y: variables (the ones we solve for)

a: parameter (solutions depend on the parameters)

exogenous

Alt: Gauss

$$\left(\begin{array}{cc|c} 1 & 1 & 4\\ 1 & a & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 4\\ 0 & a-1 & 2 \end{array}\right)$$

$a \neq 1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 4\\ 0 & a-1 & 2 \end{array}\right)$$

$a = 1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 4\\ 0 & 0 & 2 \end{array}\right)$$

no solutions

$$\begin{aligned} x+y&=4\\ (a-1)y&=2 \end{aligned}$$

$$y = \frac{2}{a-1}$$

$$x = 4-y = 4-\frac{2}{a-1} = \frac{4(a-1)-2}{a-1} = \underline{\frac{4a-6}{a-1}}$$

$$(x,y) = \underline{\left(\frac{2}{a-1}, \frac{4a-6}{a-1}\right)} \quad (a \neq 1)$$

Alt 2: determinants

$x+y=4$
$x+ay=6$

$A=\begin{pmatrix}1&1\\1&a\end{pmatrix}$

$|A|=a-1$

(a) $|A|=0$: $a=1$

no sol's or inf. many sol's

$\left(\begin{array}{cc|c}1&1&4\\1&1&6\end{array}\right)$

$\downarrow$

$\left(\begin{array}{cc|c}1&1&4\\0&0&2\end{array}\right)$

no solutions

(b) $|A|\neq 0$: $a\neq 1$

one solution

Kramer's rule

$x=\dfrac{|A_1(\underline{b})|}{|A|}=\dfrac{4a-6}{a-1}$

$y=\dfrac{|A_2(\underline{b})|}{|A|}=\dfrac{2}{a-1}$

$A=\begin{pmatrix}1&1\\1&a\end{pmatrix}$

$|A|=a-1$

$A_1(\underline{b})=\begin{pmatrix}4&1\\6&a\end{pmatrix}$

$|A_1(\underline{b})|=4a-6$

$A_2(b)=\begin{pmatrix}1&4\\1&6\end{pmatrix}$

$|A_2(b)|=6-4=2$

Kramer's rule: A linear system $A\underline{x}=\underline{b}$ (with coeff. matrix A and column $\underline{b}$ of constants), such that A is square with $|A|\neq 0$:

$$x_1=\frac{|A_1(\underline{b})|}{|A|},\quad x_2=\frac{|A_2(\underline{b})|}{|A|},\quad \dots\quad x_n=\frac{|A_n(\underline{b})|}{|A|}$$

where $A_i(\underline{b})$ is the matrix you obtain when you replace the i'th column of A with $\underline{b}$.

Ex:

$$\begin{aligned} rx + 2y - z &= 3 \\ x + (r+1)y - z &= 3 \\ -x - 2y + rz &= 1-r \end{aligned}$$

x, y, z: variables
r: parameter

3×3 lin. sys

$$|A| = \begin{vmatrix} r & 2 & -1 \\ 1 & r+1 & -1 \\ -1 & -2 & r \end{vmatrix} = r\big((r+1)r - 2\big) - 2(r-1) - 1\big(-2 - (r+1)(-1)\big)$$

$$= r(r^2 + r - 2) - 2r + 2 + 2 - r - 1$$

$$= r(r+2)(r-1) - 3r + 3$$

$$= r(r+2)\underline{(r-1)} - 3\underline{(r-1)}$$

$$= (r-1) \cdot [r(r+2) - 3]$$

$$= (r-1)(r^2 + 2r - 3)$$

$$= (r-1)(r-1)(r+3) = (r-1)^2(r+3)$$

a) $\underline{|A| = 0}$: $(r-1)^2(r+3) = 0$
$\underline{r=1}$, $\underline{r=-3}$

$\underline{r=1}$: $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \\ -1 & -2 & 1 & 0 \end{array}\right) \begin{matrix} \\ ]-1 \\ ]+1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right) \updownarrow$

no solutions for $\underline{r=1}$

$\underline{r=-3}$: $\left(\begin{array}{ccc|c} -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right) \updownarrow \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -3 & 2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right) \begin{matrix}]3 \\ \\ ]1 \end{matrix}$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & -4 & -4 & 7 \end{array}\right)]-1 \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

no solutions for $r=-3$

(b) $|A| \neq 0$: $(r-1)^2(r+3) \neq 0$
$r \neq 1, -3$ $\rightarrow$ a unique solution

Kramer's rule:

$|A| = (r-1)^2(r+3)$

$$|A_1(\underline{b})| = \begin{vmatrix} 3 & 2 & -1 \\ 3 & r+1 & -1 \\ 1-r & -2 & r \end{vmatrix} = 3\left(r(r+1)-2\right) - 2\left(3r+1-r\right) - 1\left(-6-(r+1)(1-r)\right)$$

$$= 3(r^2+r-2) - 2(2r+1) - 1(r^2-7) = 2r^2 - r - 1$$

$$\Rightarrow x = \frac{2r^2-r-1}{(r-1)^2(r+3)} = \frac{\cancel{(r-1)}(2r+1)}{(r-1)^{\cancel{2}}(r+3)} = \frac{2r+1}{(r-1)(r+3)}, \quad r \neq 1, -3$$

Similar computations to find y, z when $r \neq 1, -3$.

3 Vector equations

Ex: $x \cdot \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + z \cdot \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ no solutions

$$\begin{pmatrix} x \\ 0 \\ 4x \end{pmatrix} + \begin{pmatrix} 3y \\ -y \\ 2y \end{pmatrix} + \begin{pmatrix} 4z \\ -z \\ 6z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x + 3y + 4z \\ -y - z \\ 4x + 2y + 6z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} x + 3y + 4z &= 3 \\ -y - z &= 1 \\ 4x + 2y + 6z &= 2 \end{aligned}$$

Gauss:

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 4 & 2 & 6 & 2 \end{array}\right) \begin{matrix} \\ \\ -4 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -10 & -10 & -10 \end{array}\right) \begin{matrix} \\ -10 \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -20 \end{array}\right)$$

no solutions

Linear combinations:

Ex: $\underline{v_1}, \underline{v_2}, \underline{v_3}$ three 3-vectors

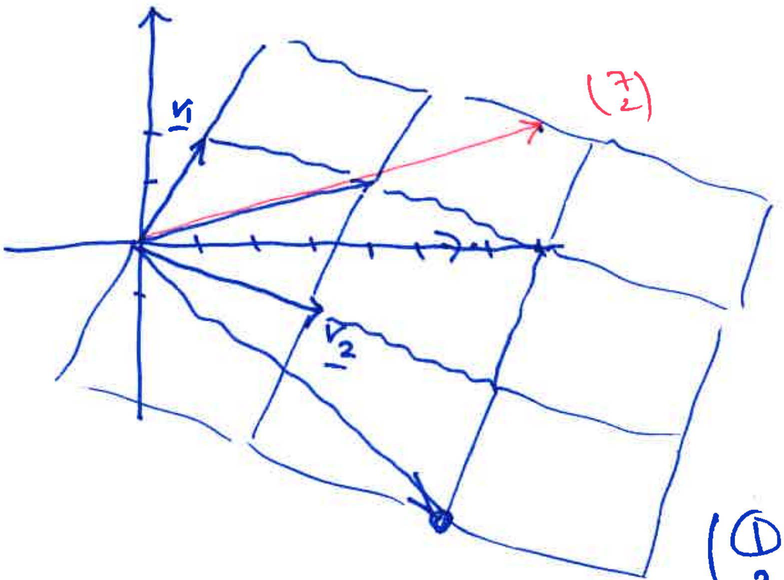
$$\underline{v_1} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \quad \underline{v_2} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \underline{v_3} = \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}$$

Defn: A linear combination of $\underline{v_1}, \underline{v_2}, \underline{v_3}$ is an expression of the form

$$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + c_3 \underline{v_3}$$

where c_1, c_2, c_3 are given numbers.

Ex: Is $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underline{v_1}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{v_2}$?



$-1 \cdot \underline{v_1} + 2\underline{v_2}$

$\underline{v_1} + \underline{v_2}$

$\underline{v_1} + 2\underline{v_2}$

intersections: $c_1 \underline{v_1} + c_2 \underline{v_2}$ where c_1, c_2 are integers

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 3 \\ -1 \end{pmatrix} :$$

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 2 \end{array}\right) \begin{matrix} \\ -2 \end{matrix} \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -12 \end{array}\right)$$

$x + 3y = 7$

$-7y = -12$

$x = 7 - 3 \cdot \frac{12}{7} = \frac{49-36}{7}$

$y = \frac{12}{7} \simeq 1.72$

$(x, y) = \underline{\left(\frac{13}{7}, \frac{12}{7}\right)} \simeq (1.86, 1.72)$