Plan

1 Matrix multiplication and linear systems
2 Computing with matrices
3 Inverse matrices

Remember: Midterm evaluation

New course paper: 14/03-21/03

1 Matrix multiplication and linear systems

Defn: A, B matrices and # cols in A = # rows in B } $A \cdot B$ matrix multiplication of A by B

↑

rows = # rows in A

cols = # cols in B

Ex: $\underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} 3 \\ 4 \end{pmatrix}} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$ $1\cdot 3 + 2 \cdot 4 = 11$, $2 \cdot 3 + 1 \cdot 4 = 10$

$2\times 2 \quad = \quad 2 \times 1 \qquad 2\times 1$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

2×2 —— 2×2 $\qquad 2 \times 2$

Note: $\boxed{AB \neq BA}$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ not defined

$2\times 1 \neq 2 \times 2$

Formula for $A \cdot B$:

If $A \cdot B$ is defined, with $A = (a_{ij})$, $B = (b_{ij})$ then:

$AB = C = (c_{ij})$ with

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

Linear systems

Ex:
$$\begin{aligned} x+y+z+w &= 4 \\ x-y+\quad 2w &= 7 \\ 2x+3y-z \quad &= 10 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 4 \\ 7 \\ 10 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

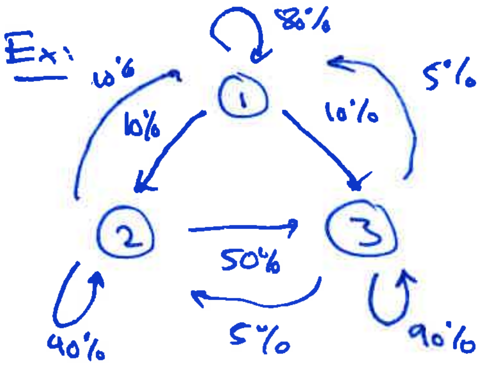
coeff. matrix

$$A \cdot \underline{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z+w \\ x-y \quad +2w \\ 2x+3y-z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 10 \end{pmatrix} = \underline{b}$$

That is; the linear system can be written $\boxed{A \cdot \underline{x} = \underline{b}}$

(matrix form of the lin. sys.)

Ex:



Starting state:

1	:	40%
2	:	30%
3	:	30%

$$A = \begin{pmatrix} 0.80 & 0.10 & 0.05 \\ 0.10 & 0.40 & 0.05 \\ 0.10 & 0.50 & 0.90 \end{pmatrix}$$

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$: after one period $\rightarrow$ $A \cdot \underline{v} = \begin{pmatrix} 0.80 & 0.10 & 0.05 \\ 0.10 & 0.40 & 0.05 \\ 0.10 & 0.50 & 0.90 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$= \begin{pmatrix} 0.80v_1 + 0.10v_2 + 0.05v_3 \\ 0.10v_1 + 0.40v_2 + 0.05v_3 \\ 0.10v_1 + 0.50v_2 + 0.90v_3 \end{pmatrix}$$

② Matrix algebra: Computing with matrices

Operations:

1) Addition, subtraction: $A+B$, $A-B$
2) Scalar multiplication: $c \cdot A$ (c: a number)
3) Matrix multiplication: $A \cdot B$
4) Powers: A^n defined when A is square and $n = 1, 2, 3, \ldots$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ not defined

$2 \times 3 \neq 2 \times 3$

$\boxed{A^0 = I}$ (identity matrix)

Special matrices: Identity matrix

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, ...

identity matrix (of size $n \times n$)

Property: $A \cdot I = A$ for any matrix A
$I \cdot A = A$

Ex: $\underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} \cdot \underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$

$\underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot \underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} = \underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$

The transpose: $A \rightsquigarrow A^T$

A: $m\times n$ matrix; A^T: $n \times m$ matrix, **transpose of A**

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ (2×3) : $A^T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$ (3×2)

$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 7 & 3 \end{pmatrix}$ (3×3) $\quad A^T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 7 \\ 0 & 4 & 3 \end{pmatrix}$

Defn: A is a **symmetric matrix** if $A^T = A$.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{pmatrix}$ are **symmetric**

$\parallel \qquad \parallel$

$A^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{pmatrix}$

Rules in matrix algebra

$A + B = B + A$

$A \cdot (B+C) = AB + AC$

$A(BC) = (AB)C$

$\vdots$

$AB \neq BA$

$(A+B)^2$

$= (A+B)(A+B)$

$= A^2 + BA + AB + B^2$

$\neq A^2 + 2AB + B^2$

Determinant:

$|A \cdot B| = |A| \cdot |B|$ ✓

$|c \cdot A| = c^n \cdot |A|$ (A is $n\times n$)

$|A^T| = |A|$

Transpose:

$(A^T)^T = A$

$(AB)^T = B^T \cdot A^T$ ✓

$x \neq 0$ number: $x^{-1} = \frac{1}{x}$ $\quad x \cdot x^{-1} = x \cdot \frac{1}{x} = 1$

③ Inverse matrices

Defn: A $n \times n$ matrix. An inverse of A is a matrix A^{-1} such that

$$A \cdot A^{-1} = I \quad \text{and}$$
$$A^{-1} \cdot A = I$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \underbrace{\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|A| = 4 - 1 = 3 \neq 0$

The case $n=2$:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- $|A| = ad - bc \neq 0$: $\quad A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- $|A| = ad - bc = 0$: no inverse of A

General facts:

i) The inverse A does not always exist. In fact, A is invertible (A^{-1} exists) if and only if $|A| \neq 0$

ii) If A has an inverse, then it is unique.

iii) General formula for A^{-1} when $|A| \neq 0$:

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

Example:

$2x + y = 4$
$x + 2y = 3$

matrix form $\longrightarrow$ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ works if A^{-1} exists

$A \cdot \underline{x} = \underline{b} \quad | \, A^{-1} \cdot$

$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$

$I \cdot \underline{x} = A^{-1} \cdot \underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

$\underline{x} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix}$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad |A| = \underline{+1 \cdot 6} - \underline{1 \cdot 5} + \underline{1 \cdot 1} = 2 \neq 0$$

$$\underline{A^{-1}} = \frac{1}{2} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T$$

$$\underline{\underline{A^{-1} = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

$C_{11} = +6$ $\qquad C_{12} = -5$ $\qquad C_{13} = +1$

$C_{21} = -6$ $\qquad C_{22} = +8$ $\qquad C_{23} = -2$

$C_{31} = +2$ $\qquad C_{32} = -3$ $\qquad C_{33} = +1$