Plan

1 Summary: Matrices, vectors and linear systems
2 Exam MET1180 05/2019 Q1, 05/2017 Q1.
3 Partial derivatives of functions in two variables

① Summary:

Linear system ($m \times n$):

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$(A|\underline{b})$ extended matrix

$A \cdot \underline{x} = \underline{b}$ matrix form

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$m \times n$ - matrix

Methods for solving linear systems:

① Gaussian elimination $(A|\underline{b}) \to \dots \to$ echelon form

② Determinants (if $m=n$):

i) $|A| \neq 0 \iff$ unique solution
ii) $|A| = 0 \iff$ no solutions or inf. many sol's

i) $|A| \neq 0$: How to find the unique solution

$A\underline{x} = \underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

or

Kramer's rule: ← especially useful when the system has parameters

$$x_1 = \frac{|A_1(\underline{b})|}{|A|}, \quad x_2 = \frac{|A_2(\underline{b})|}{|A|}, \quad \dots$$

where $A_i(\underline{b})$ is the matrix obtained from A by replacing col $\#i$ with $\underline{b}$.

Vectors:

$\underline{w}$ is a linear combination of the vectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ $\Leftrightarrow$

$\underline{w} = x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + \dots + x_n \cdot \underline{v_n}$

vector equation

$\Updownarrow$

linear system:

$$\left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \dots \,\middle|\, \underline{v_n} \right) \cdot \underline{x} = \underline{w}$$

Matrices:

- elementary row operations
- computing determinants $\leftarrow$

Methods:

- cofactor expansion
- using Gauss

$A \to \dots \to E$

i) can compute $|A|$ in terms of $|E|$

ii) $|E|$ = product of diagonal entries

Note: $|A \cdot B| = |A| \cdot |B|$

$|A| \neq 0 \Leftrightarrow A^{-1}$ exists

$AB \neq BA$

- matrix multiplication: $A \cdot B$
- inverse matrices: A^{-1}

$$|A| \neq 0 \Rightarrow A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{|A|} \begin{pmatrix} C_{11} & \dots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nn} \end{pmatrix}^T$$

$$|A| = 0 \Rightarrow A^{-1} \text{ does not exist}$$

Formulas:

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

$(AB)^T = B^T \cdot A^T$

$|A^{-1}| = \frac{1}{|A|}$

Alternative method for finding A^{-1}:

A: $n \times n$ matrix

$$(A \,|\, I) \to \dots \to (B \,|\, C)$$

reduced echelon form

$B = I$: $A^{-1} = C$

$B \neq I$: A^{-1} does not exist

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$|A| = 2\cdot 2 - 1\cdot 1 = 3 \neq 0$

$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$

Alt. method:

$(A \mid I) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right) \cdot (-2)$

$\to \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right) : (-3) \to \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right) \cdot (-2)$

echelon form

$\to \left(\begin{array}{cc|cc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right) = (I \mid A^{-1})$ $A^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$

reduced echelon form

Why does this work?

$A = \begin{pmatrix} 1 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{-2} B = \begin{pmatrix} 1 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

$B = E_1 \cdot A$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

elementary matrix

$(A \mid I) \to \dots \to (B \mid C)$

$\Updownarrow$

$E_r \cdots E_2 E_1 A = B$

$E_r \cdots E_2 E_1 \cdot I = C$

If $B = I$:

$(E_r \cdot E_{r-1} \cdots E_1) \cdot A = I$

$E_r \cdot E_{r-1} \cdots E_1 = C$

If $B \neq I$: $|A| = 0 \Rightarrow$ no inverse

2) Exam 05/2017 Q1

$A\underline{x} = \underline{b}$ with $A = \begin{pmatrix} 1+a & 2 & 1-a \\ 2 & 1+a & 2 \\ 1-a & 2 & 1+a \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 3+a \\ a^2 \\ 3-a \end{pmatrix}$

a) Solve for the case $a=1$:

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 0 & 2 & -3 \\ 0 & 2 & 2 & 2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & -3 \end{array}\right)$$

echelon form

$$\begin{aligned} 2x + 2y \quad &= 4 \\ 2y + 2z &= 2 \\ 2z &= -3 \end{aligned}$$

$z = -3/2$

$2y + 2(-3/2) = 2$

$2y = 2 + 3 = 5$

$y = 5/2$

$2x + 2(5/2) = 4$

$2x = 4 - 5 = -1$

$x = -1/2$

One solution: $(x,y,z) = (-1/2, 5/2, -3/2)$

b) $a=1$: $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$

$|A| = 2\cdot(4-4) - 2\cdot(2\cdot 2 - 0) = -8 \neq 0$

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{-8} \cdot \begin{pmatrix} 0 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 0 \end{pmatrix}^T$$

$$= \frac{1}{-8} \begin{pmatrix} 0 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Use A^{-1} to solve $A\underline{x}=\underline{b}$ when $\underline{a=1}$.

$$A\underline{x}=b$$

$$\cancel{A^{-1}}\cancel{A}\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\cdot\underline{b}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}}_{\underline{b}}$$

$$= \frac{1}{2}\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1/2 \\ 5/2 \\ -3/2 \end{pmatrix}}}$$

c) $\underline{\text{Determine } a}$ s.t. $A\underline{x}=\underline{b}$ has a unique sol'n.

$|A| \neq 0 \Rightarrow A\underline{x}=\underline{b}$ has a unique solution

$$|A| = \begin{vmatrix} 1+a & 2 & 1-a \\ 2 & 1+a & 2 \\ 1-a & 2 & 1+a \end{vmatrix} = -2\cdot(2(1+a)-2(1-a)) + (1+a)\cdot[\underline{(1+a)^2}-(1-a)^2] - 2\cdot(2(1+a)-2(1-a))$$

$$= (1+a)\cdot((\cancel{1}+2a+\cancel{a^2}) - (\cancel{1}-2a+\cancel{a^2}))$$

$$- 2\cdot 2(\cancel{2}+2a-\cancel{2}+2a)$$

$$= (1+a)\cdot(\underline{4a}) - 4\cdot\underline{4a} = 4a(1+a-4)$$

$$= \underline{4a(a-3)}$$

$|A|=0$: $4a(a-3)=0$ $\quad \underline{a=0}$ or $\underline{a=3}$

$|A|\neq 0$: $a\neq 0, a\neq 3$

$\underline{\text{Concl:}}$ one unique solution when $\underline{\underline{a\neq 0,\ a\neq 3}}$

d) Determine a such that $A\underline{x}=\underline{b}$ has <u>no solution</u>

$a\neq 0, a\neq 3$: one solution

$\underline{a=0}$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 3 \end{array}\right) \begin{matrix} -2 \\ \\ \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -3 & 2 & -6 \\ 0 & 2 & 1 & 3 \end{array}\right) \begin{matrix} \\ -1 \\ \\ \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

<u>Inf. many solutions</u>

no sol'n or inf. many sol'n.

$\underline{a=3}$:

$$\left(\begin{array}{ccc|c} 4 & 2 & -2 & 6 \\ 2 & 4 & 2 & 9 \\ -2 & 2 & 4 & 0 \end{array}\right) \begin{matrix} 1:2 \\ \\ \\ \end{matrix} \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 2 & 4 & 2 & 9 \\ -2 & 2 & 4 & 0 \end{array}\right) \begin{matrix} -1 \\ \\ 1 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & 3 & 3 & 3 \end{array}\right) \begin{matrix} \\ \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

<u>no solutions</u>

<u>Concl:</u> No solutions $\Leftrightarrow$ $\underline{\underline{a=3}}$

3 Partial derivatives

Ex: $f(x,y) = 3x + 4y - 5$

$f(x,y) = x^2 + y^2$

$f(x,y) = x^3 - 3xy + y^3$

Partial derivatives:

$$f'_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

think of y as a constant, use derivation rules to find f'_x with respect to x

$$f'_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Similar, but think of x as a constant, take derivative w.r.t. y

Ex:

$f(x,y) = 3x + 4y - 5$

$f'_x = 3$

$f'_y = 4$

Ex: $f = x^2 + y^2$

$f'_x = 2x$

$f'_y = 2y$

Ex: $f = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y(1) + 0 = 3x^2 - 3y$

$f'_y = 0 - 3x \cdot 1 + 3y^2 = -3x + 3y^2$