Plan
1 Functions in two variables
2 Graphs and level curves
3 Linear functions

Reminder:

Deadline course paper Mon at 12.00.

① Functions in two variables

Ex: $f(x,y) = 2x+3y-1$ linear fn.

$f(x,y) = x^2+y^2$ polynomial fn.

$f(x,y) = \dfrac{x+y}{x-y}$ rational fn.

$f(x,y) = x \cdot e^y$

In general: $f(x,y) =$ functional expression (expression in x,y)

(x,y): input variables

$z = f(x,y)$: output variable

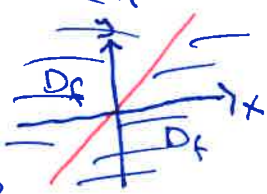
Defn: D_f = domain of f = all coordinate pairs (x,y) that we can use as inputs for the function f.

(subset of $\mathbb{R}^2$, the xy-plane)

Ex: $f(x,y) = 2x+3y-1$ D_f: any (x,y) in $\mathbb{R}^2$ $(D_f = \mathbb{R}^2)$

$f(x,y) = \dfrac{x+y}{x-y}$ D_f: $x-y \neq 0$

$D_f = \{(x,y) \in \mathbb{R}^2 : x-y \neq 0\}$



Defn: V_f = range of f = all values $f(x,y)$ we can get when (x,y) is in D_f

To find the range V_f, you have to find the max/min of f.

Ex: $f(x,y) = 2x+3y-1$ $\quad V_f = (-\infty, \infty) = \mathbb{R}$

$f(x,y) = x^2+y^2$ $\quad V_f = [0, \infty)$

② Graphs and level curves

Defn: The graph of a function f in two variables is the set of all points

$$(x,y,z)$$

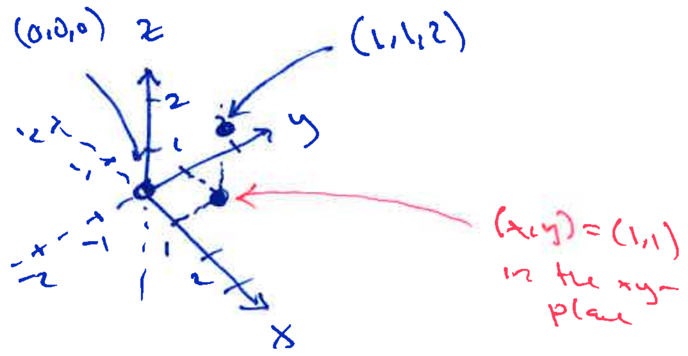
such that (x,y) is in D_f and $z = f(x,y)$.

We can draw the graph of f in the xyz-coordinate system.

Ex: $f(x,y) = x^2+y^2$, $D_f = \mathbb{R}^2$

(x,y)	$(0,0)$	$(1,0)$	$(1,1)$
$z=f(x,y)$	$0 = f(0,0)$	$1 = f(1,0)$	$2 = f(1,1)$

$f(0,0) \to (0,0,0)$ $\quad f(1,1) \to (1,1,2)$

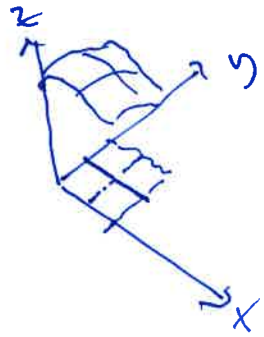


Graph of f = surface

<u>Level curves:</u> $f(x,y) = c$

for a constant c

<u>Graph of f</u>

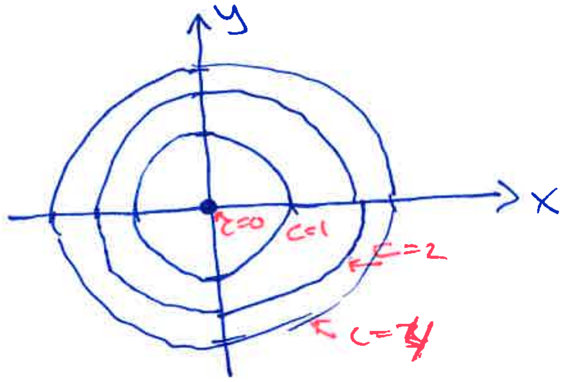


<u>Ex:</u> $f(x,y) = x^2 + y^2$

<u>$c=2$:</u> $f(x,y) = 2$

$x^2 + y^2 = 2$

circle, center $(0,0)$, $r = \sqrt{2}$



<u>$c=1$:</u> $x^2 + y^2 = 1$

circle, $r = 1$

<u>$c=0$:</u> $x^2 + y^2 = 0$

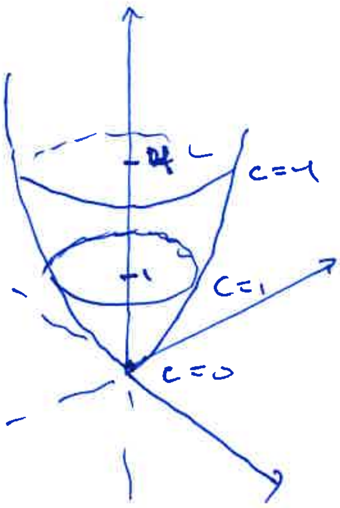
point $(0,0)$

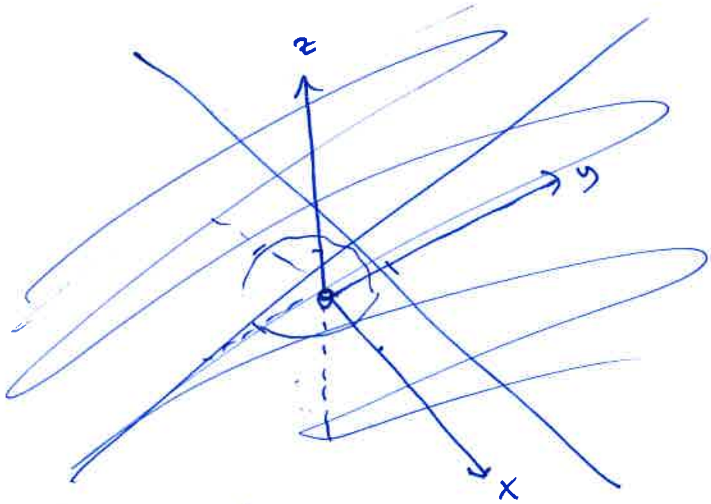
<u>$c=4$:</u> $x^2 + y^2 = 4$

circle, $r = 2$

<u>$c=-1$:</u> $x^2 + y^2 = -1$

no points



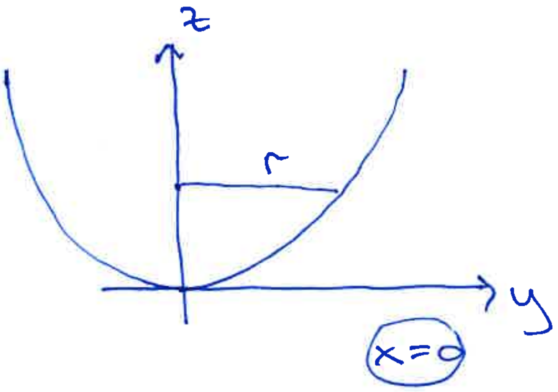


easier to draw the graph without the coordinate system - and add it in later

Cut $x=0$:

$z = f(0,y) = 0^2 + y^2$

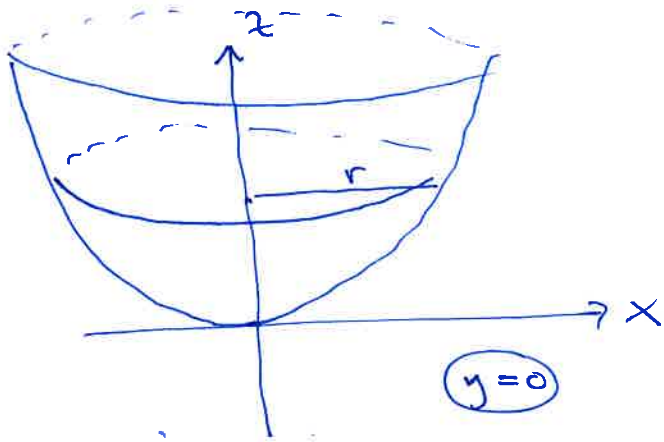
$z = y^2$



Cut $y=0$:

$z = f(x,0) = x^2 + 0^2$

$z = x^2$



Ex: $f(x,y) = x^2 - y^2$

Level curves:

$c=0$: $x^2 - y^2 = 0$

$(x-y)(x+y) = 0$

$y = x$ or $y = -x$

$c=1$: $x^2 - y^2 = 1$

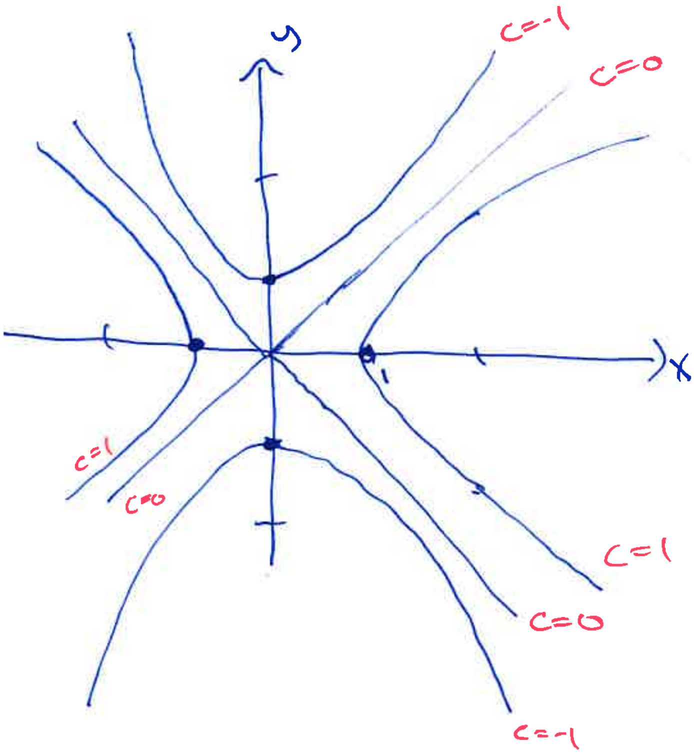
$x^2 - 1 = y^2$

$y = \pm\sqrt{x^2-1}$ $x \geq 1$ or $x \leq -1$

$c=-1$: $x^2 - y^2 = -1$

$x^2 + 1 = y^2$

$y = \pm\sqrt{x^2+1}$



③ Linear functions

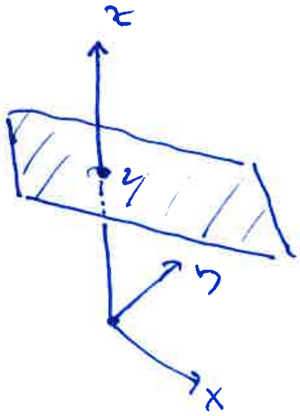
Defn: A function f in two variables is called linear if it can be written

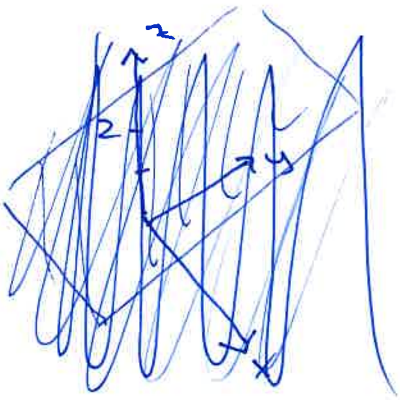
$$f(x,y) = ax + by + c$$

Fact:

The graph of f is a plane $\Longleftrightarrow$ f is linear

Ex: $f(x,y) = 2$





In general: The intersection of the graph of $f(x,y) = ax + by + c$ with the z-axis is $\underline{z = c}$.

$f(0,0) = c$ and z-axis: $x = y = 0$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ Which vectors $\underline{w}$ satisfies $\underline{v} \perp \underline{w}$?

$\underline{v} \cdot \underline{w} = 0 \Leftrightarrow \underline{v} \perp \underline{w}$

$\underline{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$: $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$1 \cdot a + (-1) \cdot b + 2c = 0$

$\underline{a - b + 2c = 0}$

$(1 \;\; -1 \;\; 2 \mid 0)$

b, c: free variables

$a - b + 2c = 0 \Rightarrow a = b - 2c$

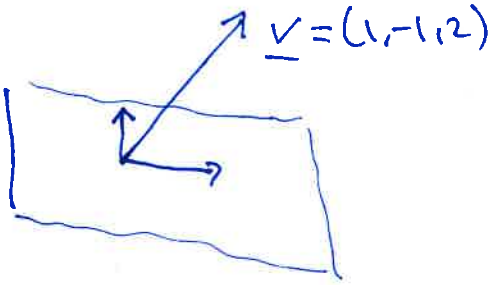
$\underline{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b - 2c \\ b \\ c \end{pmatrix}$

$= \begin{pmatrix} b \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} -2c \\ 0 \\ c \end{pmatrix}$

$= b \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

The vectors that are normal (90°) to $\underline{v}$:

All linear combinations of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$



Inner product of vectors:

$\underline{v}, \underline{w}$: n-vectors

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

Inner product = dot product:

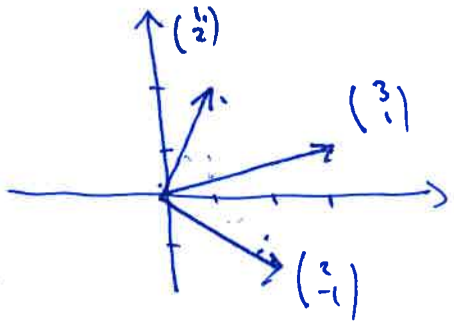
$$\underline{v} \cdot \underline{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \ldots + v_n \cdot w_n$$

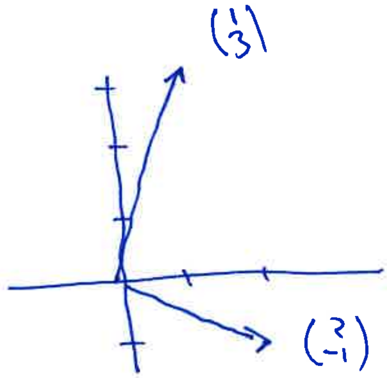
Ex:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 1 = 5$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 1 \cdot 2 + 2 \cdot (-1) = 0$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 1 \cdot 2 + 3 \cdot (-1) = -1$$





In general:

$\underline{v} \cdot \underline{w} = 0 \iff \underline{v} \perp \underline{w}$ (angle between vectors is 90°)

$\underline{v} \cdot \underline{w} > 0 \iff$ angle of $< 90^\circ$

$\underline{v} \cdot \underline{w} < 0 \iff$ angle of $> 90^\circ$

Linear function with $c=0$:

$f(x,y) = ax + by$

$z = ax + by$

$$\boxed{0 = ax + by - z} \iff \begin{pmatrix} a \\ b \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

The graph of $f(x,y) = ax+by$: All vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that are ⊥ on the fixed vector $\begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$

A plane, and $(a,b,-1)$ is called its normal vector

Ex: $f(x,y) = x - 2y$

$z = x - 2y$

$0 = x - 2y - z$

normal vector: $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

Conclusion:

The graph of $f(x,y) = ax+by+c$ is a plane with normal vector $\underline{n} = (a,b,-1)$ and intersection with the z-axis in $z=c$