EBA2910, lecture 27, 23 March 2022, Rumar Ile

1. Partial derivatives and the Hessian
2. Tangents of level curves
3. The gradient and directional derivatives

1. Partial derivatives

A function $z = f(x,y)$

Definition $f'_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ $\begin{cases} x \text{ varies} \\ y \text{ is constant} \end{cases}$

$$f'_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \quad \begin{cases} x \text{ is constant} \\ y \text{ varies} \end{cases}$$

More notation: $\dfrac{dz}{dx} = f'_x = f'_x(x,y)$, $\dfrac{dz}{dy} = f'_y = f'_y(x,y)$

Examples

i) $f(x,y) = x^2 - 4x + y^2 + 2y$ gives $f'_x = 2x - 4$
and $f'_y = 2y + 2$

ii) $f(x,y) = x^3 - 3xy + y^3$ gives $f'_x = 3x^2 - 3y$
and $f'_y = -3x + 3y^2$

iii) $f(x,y) = \sqrt{x^2+y^2}$ gives $f'_x = \dfrac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}x$

$= \sqrt{u} = u^{0,5}$ $\qquad = \dfrac{x}{\sqrt{x^2+y^2}}$

with $u = x^2 + y^2$

$(u^{0,5})'_u = 0,5\, u^{0,5-1} = \dfrac{1}{2\sqrt{u}}$ and $f'_y = \dfrac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y$

$u'_x = 2x$ and $u'_y = 2y$ $\qquad = \dfrac{y}{\sqrt{x^2+y^2}}$

Interpretation of partial derivatives

What does $f'_x(a,b)$ and $f'_y(a,b)$ mean?

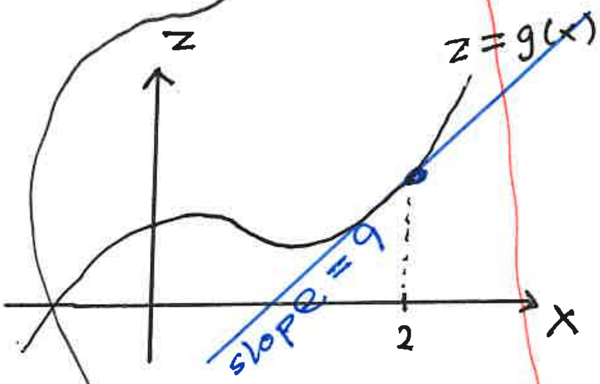
Ex $f(x,y) = x^3 - 3xy + y^3$, we had $f'_x = 3x^2 - 3y$

and $f'_y = -3x + 3y^2$

Say $(x,y) = (2,1)$, then $f(2,1) = 2^3 - 3\cdot 2\cdot 1 + 1^3 = 3$

and $f'_x(2,1) = 3\cdot 2^2 - 3\cdot 1 = 9$

and $f'_y(2,1) = -3\cdot 2 + 3\cdot 1^2 = -3$



In the x-direction ($y=1$)

From $f(x,y)$ we get a function in 1 variable

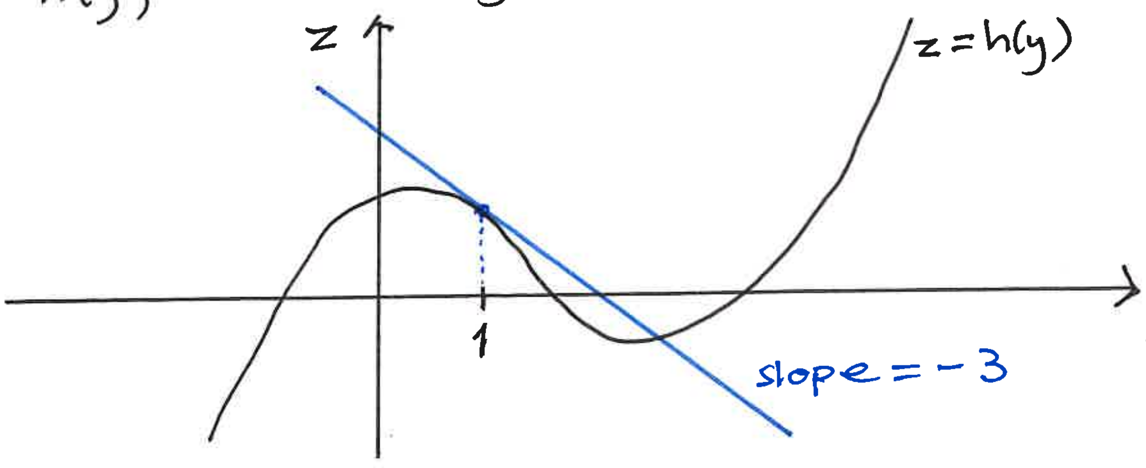
$g(x) = f(x,1) = x^3 - 3x + 1$

$g'(x) = 3x^2 - 3$ and $g'(2) = 3\cdot 2^2 - 3 = 9$

In the y-direction ($x=2$)

We get new 1-variable function $h(y) = f(2,y) = 8 - 6y + y^3$

Then $h'(y) = -6 + 3y^2$ so $h'(1) = -6 + 3 = -3$



Definition We have a function $f(x,y)$. Then a point $(x,y) = (a,b)$ is a **stationary point** for f if

$$f'_x(a,b) = 0 = f'_y(a,b).$$

How to find the stationary points? Solve the system of equations

$$\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases}$$

The Hessian of $f(x,y)$.

Definition The Hessian of $f(x,y)$ is the 2×2-matrix

$$H(f)(x,y) = \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix}$$

Ex $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y$ gives $f''_{xx} = 6x$ and $f''_{xy} = -3$

$f'_y = 3y^2 - 3x$ gives $f''_{yx} = -3$ and $f''_{yy} = 6y$

so $H(f)(x,y) = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$ and we can insert a point: $(x,y) = (1,1)$ and get $H(f)(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

{start : 11.00}

2. Tangents of level curves

<u>Ex</u> $f(x,y) = x^2 - 2x + y^2 + 4y$

The level curve $f(x,y) = c$ is the solution set of this equation.

The level curves are curves in the xy-plane.

The level curve $f(x,y) = c$ is the the "shadow" of the horizontal plane

$z = c$ intersected with the graph $z = f(x,y)$

Let see. [shows graph]

Why circles? We do completing the squares.

$$x^2 - 2x + y^2 + 4y = c$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = c + 1 + 4$$

so $$(x-1)^2 + (y+2)^2 = c+5$$

If $c+5 > 0$ that is $\underline{c > -5}$ then the level curve is a circle with $r = \sqrt{c+5}$ and centre $(x,y) = (1,-2)$

If $c+5 = 0$ i.e $\underline{c = -5}$ then $(x,y) = (1,-2)$ is the "level curve" (a point!)

If $c+5 < 0$ i.e. $c < -5$ there are no level curves.

The tangent lines to level curves.

Ex (cont'd) If $(x, y) = (-2, 2)$ then $c = 20$ since $f(-2,2) = 20$. Use the point-slope formula to write the eq:

$y - 2 = k \cdot (x+2)$ where I claim $k = 3/4$ (the slope)

$y = \frac{3}{4}(x+2) + 2$

$y = \frac{3}{4}x + \frac{7}{2}$

If $(x, y) = (1, 0)$ then $f(1,0) = -1 = c$

$y = k \cdot (x-1)$ with (claim) $k = 0$

so $y = 0$ (the x-axis)

For the claims we use implicit differentiation. Think $y = y(x)$.

$(x^2 - 2x + y^2 + 4y)'_x = (20)'$

$2x - 2 + 2yy' + 4y' = 0$ and solve

for $y' = -\frac{2x-2}{2y+4} \overset{\text{observation}}{=} -\frac{f'_x}{f'_y}$ inserted $(-2, 2)$

gives $y'|_{(-2,2)} = -\frac{2\cdot(-2)-2}{2\cdot 2+4} = -\frac{-6}{8} = \frac{3}{4}$ -ok

Result If $f(x,y) = c$ then

$$f'_x + f'_y \cdot y' = 0$$

In particular $$y' = -\frac{f'_x}{f'_y}$$

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3. The gradient

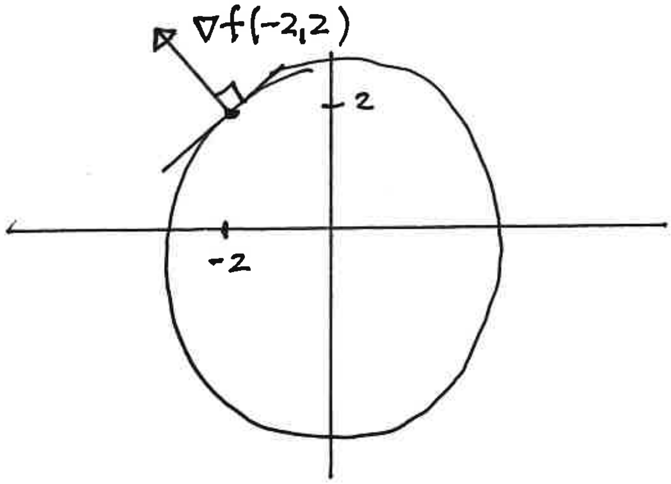
The gradient of $f(x,y)$ is the vector $\nabla f = \begin{bmatrix} f'_x \\ f'_y \end{bmatrix}$ $\left([f'_x, f'_y] \right)$

Ex $f(x,y) = x^2 - 2x + y^2 + 4y$

$$\nabla f = \begin{bmatrix} 2x - 2 \\ 2y + 4 \end{bmatrix}$$

Inserted $(x,y) = (-2,2)$ gives

$$\nabla f(-2,2) = \begin{bmatrix} 2\cdot(-2) - 2 \\ 2\cdot 2 + 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$



is a normal vector to the tangent line for the level curve at $(-2, 2)$.

Directional derivatives

Ex: If $\underline{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ then

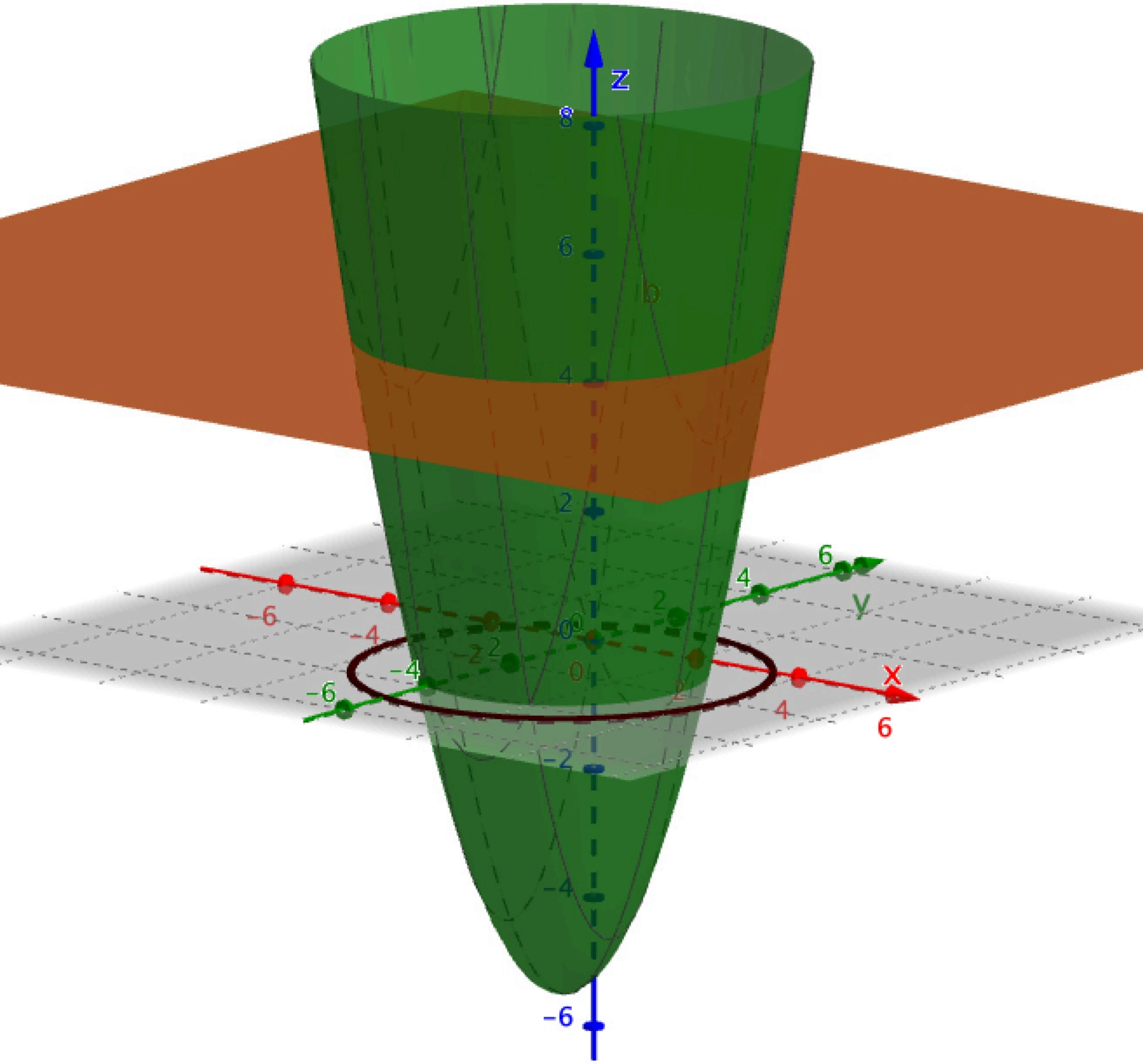
$$f'_{\underline{a}} = \underline{a} \bullet \nabla f \overset{\text{ex above}}{=} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$$

↑
inner product

$$= 2(2x-2) + 1 \cdot (2y+4)$$

$$= 4x - 4 + 2y + 4 = 4x + 2y.$$

E.g. $f'_{\underline{a}}(1,1) = 4 \cdot 1 + 2 \cdot 1 = 6$.



z
8
6
b
4
2
6
4
y
2
−6
−4
0
−2
2
−4
0
−6
2
x
4
6
−2
−4
−6