Plan

1. Stationary points
2. Second derivative test
3. Global maximum and minimum

Optimization:

max/min $f(x,y)$

Note:

maximum = global maximum
minimum = global minimum

Defn:

(x^*,y^*) is a maximal pt / maximizer for f if $f(x^*,y^*) \geq f(x,y)$ for all (x,y) in D_f

(x^*,y^*) is a local max. for f if $f(x^*,y^*) \geq f(x,y)$ for all (x,y) close to (x^*,y^*)

(x^*,y^*) is a minimum pt / minimizer for f if $f(x^*,y^*) \leq f(x,y)$ for all (x,y) in D_f

(x^*,y^*) is a local min. for f if $f(x^*,y^*) \leq f(x,y)$ for all (x,y) close to (x^*,y^*)

A stationary pt (x^*,y^*) of f that is neither local max nor local min is called a saddle point

Key result:

If (x^*,y^*) is a max/min for f, then we have either

i) (x^*,y^*) is a stationary pt for f ($f'_x = f'_y = 0$ at (x^*,y^*))

ii) either f'_x or f'_y is not defined at (x^*,y^*)

iii) (x^*,y^*) is a boundary pt for D_f.

Names: i) or ii) are called critical points

candidate pts: i), ii) or iii)

Note: If f is "nice" (for example a polynomial), then the only pts of type i) exist

Ex. $f(x,y) = x^3 + 3xy + y^3$

Crit pts = stationary pts

Stationary pts:

$f'_x = 3x^2 + 3y = 0$
$f'_y = 3x + 3y^2 = 0$
$\Big\}$ first order conditions (FOC)

$x^2 + y = 0 \Rightarrow \boxed{y = -x^2}$
$x + y^2 = 0$

$x + (-x^2)^2 = 0$

$x + x^4 = 0$

$x(1 + x^3) = 0$

$x = 0$ or $x^3 = -1$

$y = 0$ $x = \sqrt[3]{-1} = -1$

$y = -1$

Stationary pts = Crit. pts

$(x,y) = (0,0), (-1,-1)$

$f(0,0) = 0$ cand. for min

$f(-1,-1) = 1$ cand. for max

② Second derivative test:

Second derivative test

If (x^*, y^*) is a stationary pt. of f, then we compute

$$H(f)(x^*,y^*) = \begin{pmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{xy}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

We have that:

(1) If $\underbrace{\det H(f)(x^*,y^*)}_{= AC - B^2} > 0$ and $\underbrace{\operatorname{tr} H(f)(x^*,y^*)}_{= A + C} > 0$ then (x^*,y^*) is local min

(2) If $\underbrace{\det H(f)(x^*,y^*)}_{= AC - B^2} > 0$ and $\underbrace{\operatorname{tr} H(f)(x^*,y^*)}_{= A + C} < 0$ then (x^*,y^*) is local max

(3) If $\det H(f)(x^*,y^*) < 0$, then (x^*,y^*) is a saddle point.

Note: If $\det H(f)(x^*,y^*) = 0$, then the test is inconclusive.

Ex: (continued) $f(x,y) = x^3 + 3xy + y^3$

$f'_x = \underline{3x^2 + 3y}$

$f'_y = \underline{3x + 3y^2}$

Crit. pts:

$(x^*, y^*) = \underset{f=0}{\underline{(0,0)}}, \underset{f=1}{\underline{(-1,-1)}}$

We classify the stationary pts using the second derivative test: = determine (for each stationary pt) if it is local max, local min, saddle pt.

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$$

For (0,0):

$$H(f)(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\det = \det H(f)(0,0) = 0 - 9 = -9 < 0$

$\Downarrow$

$(0,0)$ is a $\underline{\underline{\text{saddle}}}$ pt for f

For (-1,-1):

$$H(f)(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$$

$\underset{\underset{\det}{\shortparallel}}{AC-B^2}$

$\det = 36 - 9 = 27 > 0$

$\underset{\underset{A+C}{\shortparallel}}{\text{tr}} = \text{tr}\, H(f)(-1,-1) = -6 + (-6) = -12 < 0$

$\Downarrow$

$(-1,-1)$ is a $\underline{\underline{\text{local max}}}$ for f

③ Global max/min

Conclusion:

$f(x,y) = x^3 + 3xy + y^3$ has $\underline{\text{no maximum}}$

— " — has $\underline{\text{local maximum}}$ $f(-1,-1) = 1$

Ad hoc: $f(10,10) = 10^3 + 3 \cdot 10 \cdot 10 + 10^3 = 1000 + 300 + 1000 = 2300 > 1$

$\Rightarrow$ $(-1,-1)$ is no max for f

$\Rightarrow$ f $\underline{\text{has no maximum}}$

Note: If (x^*, y^*) is stationary pt with

$$H(f)(x^*, y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

then:

① If $AC - B^2 > 0$, then $AC > B^2 \geq 0$ hence $AC > 0$

$\Rightarrow A, C > 0 \iff A + C > 0$

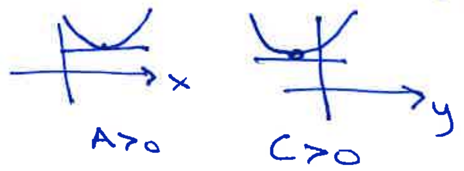
or

$A, C < 0 \iff A + C < 0$

In short, the possible cases are $\begin{cases} \text{i) } A, C > 0 \\ \text{ii) } A, C < 0 \end{cases}$

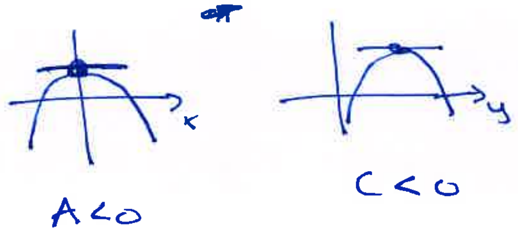
cuts of the graph $z = f(x, y)$

$A, C > 0$:



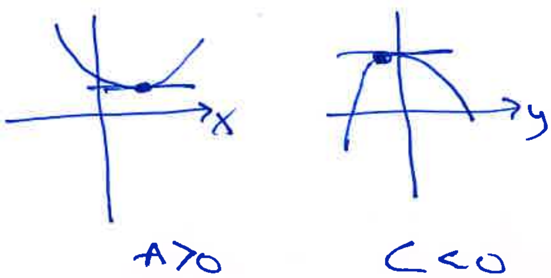
local min

$A, C < 0$:



local max

② If $AC - B^2 < 0$, a typical case is $A > 0, C < 0$



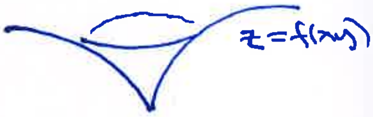
Saddle pt

Ex: $f(x,y) = \sqrt{x^2+y^2}$, $D_f = \mathbb{R}^2 \Rightarrow$ no boundary pts of D_f

$u = x^2+y^2$

$f'_x = (\sqrt{u})'_x = (u^{1/2})'_x$

$= \frac{1}{2} u^{-1/2} \cdot u'_x = \frac{1}{2\sqrt{u}} \cdot 2x$



Stat. pts:

$f'_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = 0 \Rightarrow x = 0$

$f'_y = \frac{1}{2\sqrt{u}} \cdot u'_y = \frac{y}{\sqrt{x^2+y^2}} = 0 \Rightarrow y = 0$

But $f'_x(0,0)$, $f'_y(0,0)$ gives division by 0

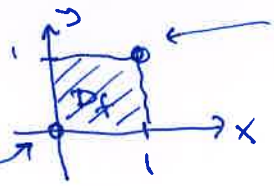
$\Rightarrow (x^*, y^*) = (0,0)$ is not stationary pt. $\Rightarrow$ no stat. pts. for f.

f'_x, f'_y not defined for $(0,0) \Rightarrow (0,0)$ is a critical pt

Note that $f(0,0) = 0$ is a minimum pt for f.

max: $f(1,1) = 2$

Ex: $f(x,y) = x+y$, $0 \le x, y \le 1$



min: $f(0,0) = 0$

Boundary pts of D_f = the sides of the square

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$ max/min $f(x,y) = ?$

Candidate pts = maximum pts:

FOC: $f'_x = 2xy^3 = 0 \implies x = 0$ or $y = 0$

$f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0$

$x = 0$	$y = 0$
$0 + 2y - 2 = 0$	$0 + 0 - 2 = 0$
$2y = 2$	impossible
$y = 1$	

$(x^*, y^*) = (0,1)$

Stat. pts: $(x^*, y^*) = (0,1)$

$f = -1$

Classification:

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y + 2 \end{pmatrix}$$

For $(0,1)$: $H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\det = 4 > 0$, $\operatorname{tr} = 4 > 0$

$\Downarrow$

$f(0,1) = -1$ is local min

Ad hoc: $x = 1$

$f(1,y) = y^3 + y^2 - 2y$

$f(1,-10) = -1000 + 100 + 20$

$= -880$

$\Downarrow$

f has no minimum (or maximum)

$y \to -\infty \implies f(1,y) \to -\infty$

Note: Example of fn. $f(x,y)$ with only one stat. pt, which is a local min, but still not global min.

Linear approximation of $f(x,y)$ at (x_0,y_0): tangent plane of f at (x_0,y_0)

$$L(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0)$$