Plan

1 Lagrange problems
2 Lagrange's multiplier method

Note: Lecture 31 Fri (~~tomorrow~~) 8-10.

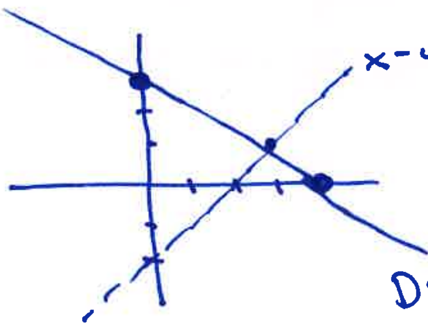
① Lagrange problems

Lagrange problems = optimization problems (max/min) with equality constraints

(*) $\boxed{\text{max/min } f(x,y) \text{ when } g(x,y) = a}$

g: function, a: constant

Ex: min $f(x,y) = x^2+y^2$ when $\underbrace{3x+4y}_{g(x,y)} = \underbrace{12}_{a}$



Ex: min $f(x,y)=x^2+y^2$
when $\left.\begin{aligned} 3x+4y&=12 \\ x-y&=2 \end{aligned}\right\}$ a pt.

$$\frac{4y}{4} = \frac{12-3x}{4}$$

$$y = 3 - \frac{3}{4}x$$

General method:

i) Find candidate points

ii) Determine whether any of these are max/min

Lagrange problems

i) ~~stationary pts that are interior pts of D~~

ii) ~~other critical pts that are interior pts of D~~

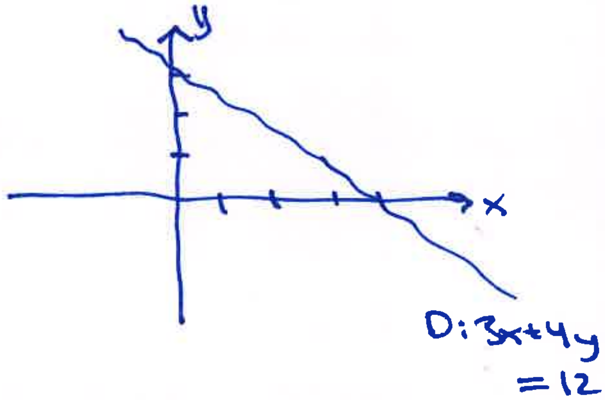
iii) boundary pts of D

Lagrange pb.

Yes

Extreme Value Thm:
If D is compact (closed and bounded) and f is cont., then f has a max/min on D

Ex: min $f(x,y) = x^2+y^2$ when $3x+4y=12$



not compact

② Method of Lagrange multipliers:

$$L(x,y;\lambda) = f(x,y) - \lambda\cdot(g(x,y)-a)$$ ← Lagrangian

Lagrange multiplier

$$= x^2+y^2-\lambda(3x+4y-12)$$

Candidates for max/min: Stationary pts of L

FOC: $\begin{cases} L'_x = f'_x - \lambda\cdot g'_x = 2x-\lambda\cdot 3 = 0 \\ L'_y = f'_y - \lambda\cdot g'_y = 2y - \lambda\cdot 4 = 0 \end{cases}$

C: $L'_\lambda = -1\cdot(g(x,y)-a) = -(3x+4y-12) = 0$ $\quad 3x+4y-12=0$

$$\boxed{3x+4y=12}$$

Lagrange conditions: FOC + C

FOC: $L'_x = 2x - 3\lambda = 0 \to 2x = 3\lambda \to x = \frac{3}{2}\lambda = \frac{3}{2}\cdot\frac{24}{25} = \frac{36}{25}$

$L'_y = 2y - 4\lambda = 0 \to 2y = 4\lambda \to y = 2\lambda = 48/25$

C: $3x + 4y = 12$

C: $3\left(\frac{3}{2}\lambda\right) + 4(2\lambda) = 12 \quad | \cdot 2$

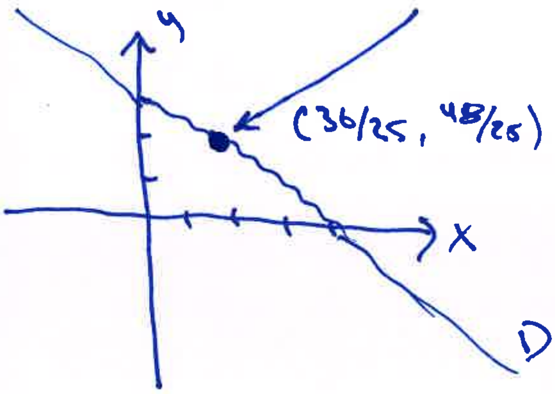
$9\lambda + 16\lambda = 24$

$25\lambda = 24$

$\lambda = 24/25$

Candidate pts:

$(x, y; \lambda) = \left(\frac{36}{25}, \frac{48}{25}; \frac{24}{25}\right)$



Alternative method: (substitution)

min $f(x,y) = x^2 + y^2$ wh $3x + 4y = 12$

$4y = 12 - 3x$; divide by 4: $y = 3 - \frac{3}{4}x$

$= x^2 + \left(3 - \frac{3}{4}x\right)^2$

$= x^2 + 9 - \frac{9}{2}x + \frac{9}{16}x^2$

$= \frac{25}{16}x^2 - \frac{9}{2}x + 9$

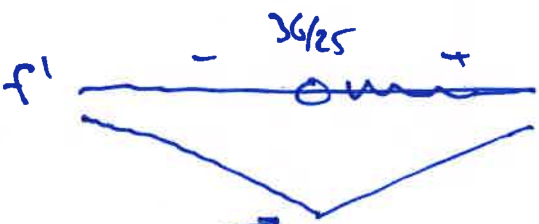
min $f(x) = \frac{25}{12}x^2 - \frac{9}{2}x + 9$

$f'(x) = \frac{25}{16}\cdot 2x - \frac{9}{2} = 0$

$\frac{25}{8}x - \frac{9}{2} = 0 \quad | \cdot 8$

$25x - 36 = 0$

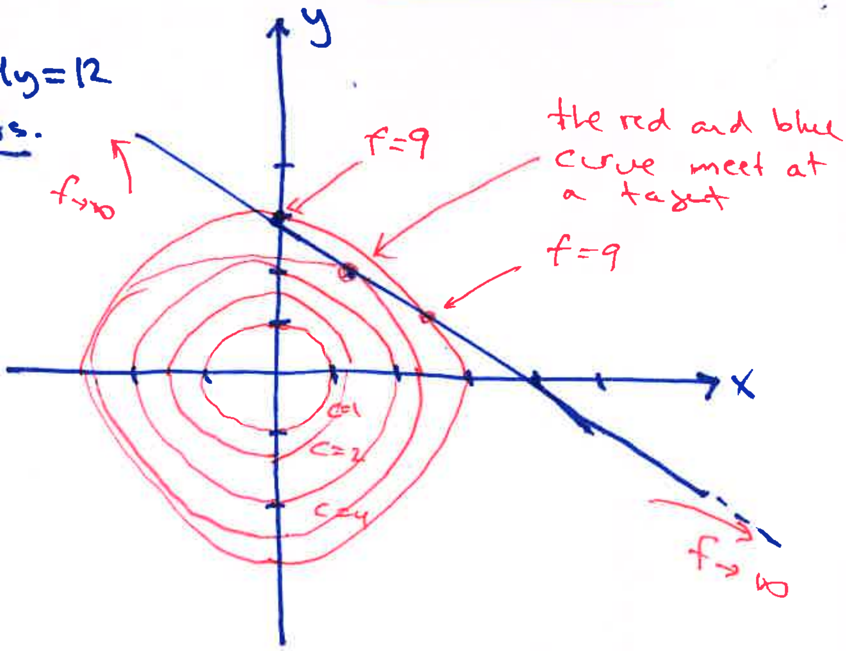
$x = 36/25$



is a minimum

Ex: max/min $f(x,y) = x^2 + y^2$ when $3x + 4y = 12$

D: $3x+4y=12$
adm. pts.



Level curves of f:

$f(x,y) = c$

$x^2 + y^2 = c$ ← circle, center $(0,0)$, $r = \sqrt{c}$ when $c > 0$; pt. $(0,0)$ if $c = 0$; no pts if $c < 0$

$c=1$: $x^2+y^2=1$

$c=2$: $x^2+y^2=2$

$c=4$: $x^2+y^2=4$

$c=9$: $x^2+y^2=9$

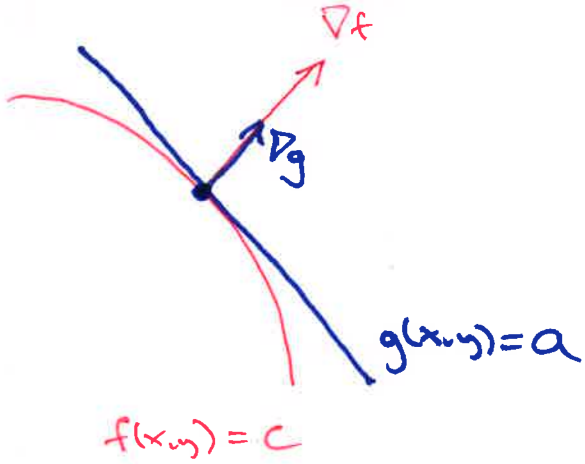
Candidates for max/min: pts where $3x+4y=12$ (D) and $x^2+y^2=c$ (level curve) meet at a tangent

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y}$$

$$+\frac{2x}{2y} = +\frac{3}{4} \quad \Big| \cdot 4y$$

$$2 \cdot 2x = 3y$$
$$4x = 3y$$
$$y = \frac{4}{3}x$$

$$3x + 4\left(\frac{4}{3}x\right) = 12 \Big| \cdot 3$$
$$9x + 16x = 36$$
$$25x = 36$$
$$x = 36/25$$



$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y} \iff \nabla f = \lambda \cdot \nabla g$$

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

$$f'_x = \lambda g'_x$$
$$f'_y = \lambda \cdot g'_y$$

$$\boxed{\begin{aligned} L'_x &= f'_x - \lambda \cdot g'_x = 0 \\ L'_y &= f'_y - \lambda \cdot g'_y = 0 \end{aligned}}$$ FOC

Theorem:

If (x^*, y^*) is max/min in a Lagrange problem

$$\boxed{\max/\min f(x,y) \quad \text{wh} \quad g(x,y)=a}$$

then either i) there is a λ such that $(x^*, y^*; \lambda)$ satisfies the Lagrange conditions FOC+C

$$\boxed{\text{FOC: } \begin{cases} L'_x = 0 \\ L'_y = 0 \end{cases} \qquad \text{C: } g(x,y)=a}$$

ordinary candidate pts

or ii) the constraint is degenerate at (x^*, y^*)

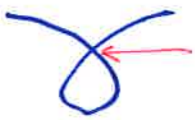
i.e.: $$\boxed{\begin{matrix} g'_x = 0 & g(x,y)=a \\ g'_y = 0 & \end{matrix}}$$

Ex: min $f = x^2 + y^2$ wh $\underbrace{3x+4y}_{g(x,y)} = 12$

$$g'_x = 3 = 0 \qquad g'_y = 4 = 0$$

not possible

Ex:



no tangent at this pt.

D

node

In general, a cand. pt. with degenerate constraint is a pt. where D does not have a unique tangent

Ex: $\max/\min\ f(x,y) = x^2+y^2$ when $xy=1$

$\max/\min\ f(x,y) = xy$ when $x^2+y^2=4$

Will solve these problems on Fri