Plan

1 Interpretation of Lagrange multipliers
2 Kuhn-Tucker problems

Ex: max/min $f(x,y)=xy$ when $x^2+y^2=1$

$\mathcal{L} = xy - \lambda(x^2+y^2-1)$

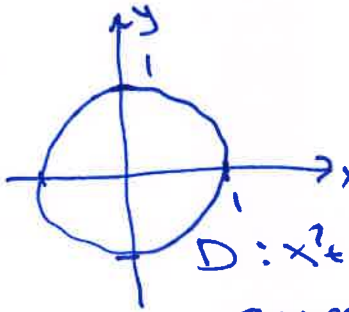
FOC $\begin{cases} \mathcal{L}'_x = y - \lambda\cdot 2x = 0 \\ \mathcal{L}'_y = x - \lambda\cdot 2y = 0 \end{cases}$

C $\quad x^2+y^2 = 1$

$y = \underline{2\lambda x}$

$x - 2\lambda(2\lambda x) = 0$

$x\cdot(1-4\lambda^2)=0$



$D: x^2+y^2=1$

compact

$\underline{x=0}$ or $\underline{\lambda^2 = 1/4}$

no adm pts. with degenerate constraint since D is a circle.

$x=0$	$\lambda = 1/2$	$\lambda = -1/2$
$y=0$	$y=x$	$y=-x$
$x^2+y^2=0 \neq 1$	$x^2+y^2=1$	$x^2+y^2=1$
no cand. pts	$2x^2=1$	$2x^2=1$
	$x^2=1/2$	$x^2=1/2$
	$x=y=\pm\sqrt{1/2}$	$x=\pm\sqrt{1/2}$, $y=-x$
	$(\sqrt{1/2},\sqrt{1/2};\frac{1}{2})$, $f=1/2$	$(\sqrt{1/2},-\sqrt{1/2};-\frac{1}{2})$, $f=-1/2$
	$(-\sqrt{1/2},-\sqrt{1/2};\frac{1}{2})$, $f=1/2$	$(-\sqrt{1/2},\sqrt{1/2};-\frac{1}{2})$, $f=-1/2$

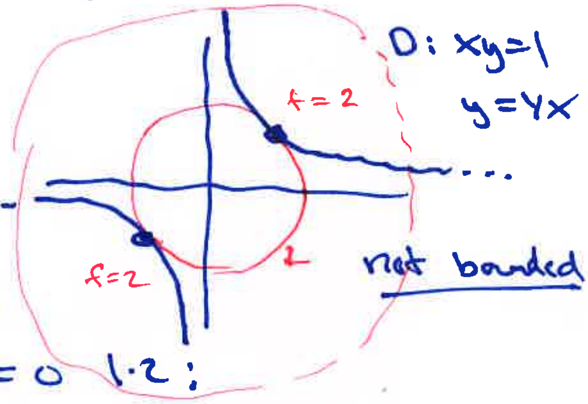
Conclusion:

D compact $\Rightarrow$ we have max/min

$f_{max} = \frac{1}{2}$ at $(\sqrt{1/2},\sqrt{1/2}), (-\sqrt{1/2},-\sqrt{1/2})$

$f_{min} = -\frac{1}{2}$ at $(\sqrt{1/2},-\sqrt{1/2}), (-\sqrt{1/2},\sqrt{1/2})$

Ex: $\max/\min\ f(x,y) = x^2+y^2$ when $xy = 1$!



$$\mathcal{L} = x^2+y^2 - \lambda(xy-1)$$

$$\text{FOC:}\begin{cases} \mathcal{L}'_x = 2x - \lambda y = 0 \\ \mathcal{L}'_y = 2y - \lambda x = 0 \end{cases}$$
$$\text{C:}\quad xy = 1$$

$\frac{2x}{2} = \frac{\lambda y}{2}$ $\quad \underline{x = \frac{\lambda y}{2}}$

$2y - \lambda\cdot\left(\frac{\lambda y}{2}\right) = 0 \quad |\cdot 2:$

$4y - \lambda^2 y = 0$

$y\cdot(4-\lambda^2) = 0$

since D is a hyperbola, there are no adm pts with degenerate constraints

$\underline{y=0}$ or	$\underline{\lambda = 2}$ or	$\underline{\lambda = -2}$
$x=0$	$x=y$	$x=-y$
$xy = 0 \neq 1$	$xy=1$	$xy=1$
<u>no cand. pts</u>	$x^2=1$	$(-y)\cdot y = 1$
	$x=\pm 1$	$y^2=-1$
	$\underline{(1,1;2)}$ $f=2$	<u>no cand pts</u>
	$\underline{(-1,-1;2)}$ $f=2$	

Level curves of f:

$f(x,y) = x^2+y^2 = c$

circle with $r=\sqrt{c}$

for $c>0$

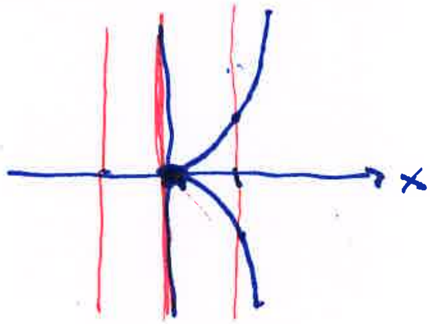
Concl: $\underline{\underline{f_{min} = 2}}$ at $(1,1), (-1,-1)$ with $\underline{\lambda = 2}$

<u>no maximum</u>

Ex: max/min $f(x,y) = x$ when $y^2 - x^3 = 0$

$\Downarrow$

$f_{min} = 0$ at $(0,0)$



Lagrange:

$\mathcal{L} = x - \lambda(y^2 - x^3)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1 + \lambda \cdot 3x^2 = 0 \\ \mathcal{L}'_y &= -\lambda \cdot 2y = 0 \\ & y^2 - x^3 = 0 \end{aligned}}$$

$-\lambda \cdot 2y = 0$

$\lambda = 0$ or $y = 0$

$1 + 0 = 0$ not possible

$0 - x^3 = 0$

$x = 0$

$1 + \lambda \cdot 0 = 0$ impossible

no ordinary candidate pts

D: $y^2 - x^3 = 0$

$y^2 = x^3$

$y = \pm\sqrt{x^3}$

$g(x,y) = y^2 - x^3$ $a = 0$

$$\boxed{\begin{aligned} g'_x &= -3x^2 = 0 \\ g'_y &= 2y = 0 \end{aligned}}$$

$x = 0$
$y = 0$

$0^2 - 0^3 = 0$

$(0,0)$ is on D.

$(0,0)$ is an adm pt with degenerate constr.

this is the minimum pt

Level curves of f:

$f(x,y) = x = c$ vertical line

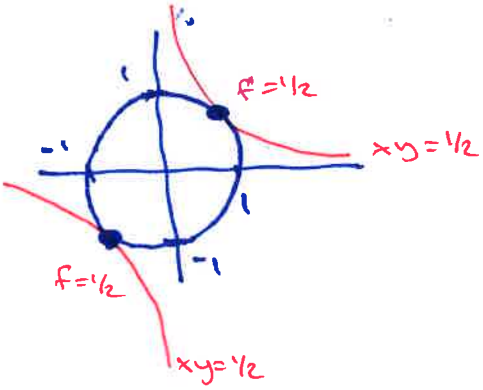
1 Interpretation of Lagrange multipliers

Ex: max/min $f(x,y) = xy$ when $x^2+y^2=1$ ← think of this as a parameter

$f_{max} = \frac{1}{2}$ at $(\sqrt{1/2}, \sqrt{1/2})$

$(-\sqrt{1/2}, -\sqrt{1/2})$

with $\lambda = 1/2$



Defn: Consider max $f(x,y)=xy$ when $x^2+y^2=a$

We write $(x^*(a), y^*(a))$ for the maximum pt, and

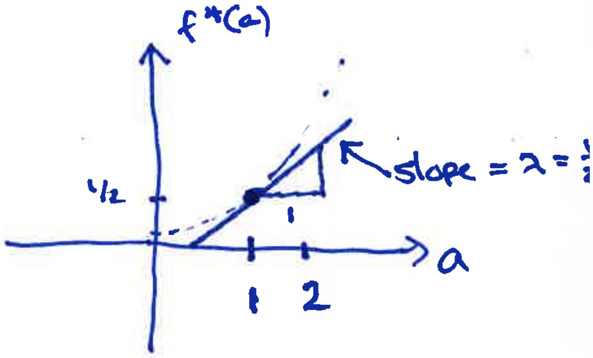
maximum value fn. → $f^*(a) = f(x^*(a), y^*(a))$ for the maximum value.

$a=1$: $x^*(1) = \sqrt{1/2}$ $y^*(1) = \sqrt{1/2}$ $f^*(1) = 1/2$

or

$x^*(1) = -\sqrt{1/2}$ $y^*(1) = -\sqrt{1/2}$ $f^*(1) = 1/2$

Result: $\lambda = \dfrac{df^*(a)}{da}$



Interpretation of λ:

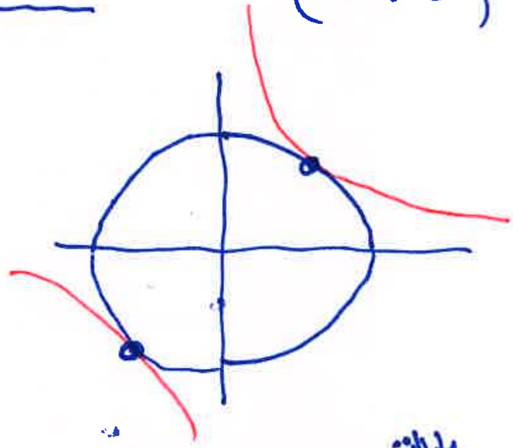
marginal change in the max. (min.) value per unit change in the constant a in the constant a in the constraint $g(x,y) = a$

$a=2$: $f^*(2) \cong f^*(1) + \Delta a \cdot \dfrac{df^*(a)}{da}$

$= \frac{1}{2} + 1 \cdot 1/2$

$= 1$

max $f(x,y) = xy$ when $\underline{x^2+y^2=a}$ $(a>0)$



$L = xy - \lambda(x^2+y^2-a)$

$L'_x = y - \lambda \cdot 2x = 0$ $\quad$ $\underline{y = 2\lambda x}$

$L'_y = x - \lambda \cdot 2y = 0$ $\quad$ $x - 4\lambda^2 x = 0$

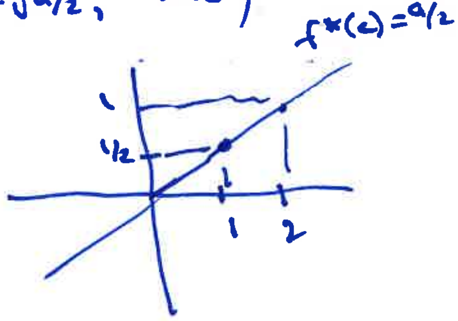
$x^2+y^2 = a$ $\quad$ $x(1-4\lambda^2)=0$

$\underline{x=0}$ or $\underline{\lambda^2 = 1/4}$

$\underline{x=0}$	$\underline{\lambda = 1/2}$	$\underline{\lambda = -1/2}$
$y=0$	$y=x$	$y=-x$
$0^2+0^2=0 \neq a$	$2y^2=a$	$2y^2=a$
$\underline{\text{impossible}}$	$y^2 = a/2$	$y^2=a/2$
	$x=y=\pm\sqrt{a/2}$	
	$\underline{(\sqrt{a/2},\sqrt{a/2};\ \tfrac{1}{2})}$	$\underline{(\sqrt{a/2},-\sqrt{a/2};\ -1/2)}$
	$f = a/2$	$f=-a/2$
	$\underline{(-\sqrt{a/2},-\sqrt{a/2};\ \tfrac{1}{2})}$	$\underline{(-\sqrt{a/2},\sqrt{a/2};\ -1/2)}$
	$f=a/2$	$f=-a/2$

$\underline{\text{Concl:}}$ $f_{max} = \underline{\underline{a/2}}$ at $(\sqrt{a/2},\sqrt{a/2})$
$(-\sqrt{a/2},-\sqrt{a/2})$

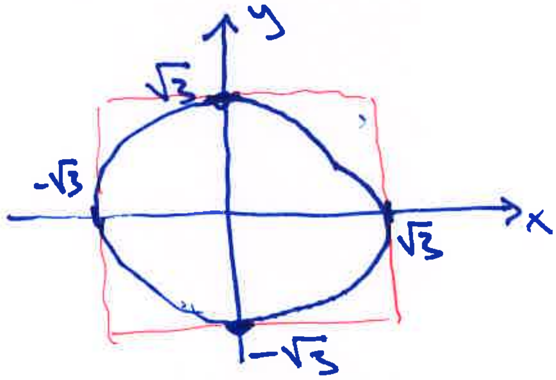
$f^*(a) = \underline{\underline{a/2}}$



Ex: $\max f(x,y) = x^2y^2$ when $x^2+y^2+x^2y^2 = 3$

$L = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$

$$\boxed{\begin{aligned} L'_x &= 2xy^2 - \lambda(2x+2xy^2) = 0 \\ L'_y &= x^2\cdot 2y - \lambda(2y + x^2\cdot 2y) = 0 \\ & x^2+y^2+x^2y^2 = 3 \end{aligned}}$$



$x=0$ or $(\ldots)=0$

(1) $2x(y^2 - \lambda - \lambda y^2) = 0$

(2) $2y(x^2 - \lambda - \lambda x^2) = 0$

$y=0$ or $(\ldots)=0$

$\Downarrow$

(a) $x=0,\ y=0$:

$0+0+0=3$ impossible

(b) $x=0,\ x^2-\lambda-\lambda x^2=0$: $x=0,\ \lambda=0$

$0+y^2+0=3 \Rightarrow y=\pm\sqrt{3}$

$(0,\pm\sqrt{3};0)$ $f=0$

(c) $y=0,\ y^2-\lambda-\lambda y^2=0$: $y=0,\ \lambda=0$

$x^2+0+0=3 \Rightarrow x=\pm\sqrt{3}$

$(\pm\sqrt{3},0;0)$ $f=0$

(d) $y^2-\lambda-\lambda y^2=0,\ x^2-\lambda-\lambda x^2=0$

$y^2 = \lambda(1+y^2) \Rightarrow \lambda = \frac{y^2}{1+y^2}$

$x^2 = \lambda(1+x^2) \Rightarrow \lambda = \frac{x^2}{1+x^2}$

$\Rightarrow \frac{y^2}{1+y^2} = \frac{x^2}{1+x^2} \quad | \cdot (1+x^2)(1+y^2)$

$\rightarrow y^2\cdot(1+x^2) = x^2(1+y^2)$

$y^2 + \cancel{x^2y^2} = x^2 + \cancel{x^2y^2}$

$y^2 = x^2 \Rightarrow y = \pm x$

$x^2+x^2+x^2\cdot x^2 = 3$

D: $x^2+y^2+x^2y^2 = 3$

Is D bounded? Yes

$x^2 \le 3 \quad -\sqrt{3} \le x \le \sqrt{3}$

$y^2 \le 3 \quad -\sqrt{3} \le y \le \sqrt{3}$

$\Downarrow$

there is a max

$x^4 + 2x^2 - 3 = 0$ quadr. eqn. in x^2

$x^2 = \dfrac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-3)}}{2} = \dfrac{-2 \pm 4}{2} = -3, 1$

$\cancel{x^2 = -3}$ or $x^2 = 1 \Rightarrow x = \pm 1$ $\quad y = \pm x = \pm 1$

$x = \pm 1$ $\quad \lambda = \dfrac{x^2}{x^2+1} = \dfrac{1}{2}$

$\Downarrow$

$\underline{(\pm 1, \pm 1; \tfrac{1}{2})}$ $f = 1$

<u>Adm pts with degenerate constr:</u> $g(x,y) = x^2 + y^2 + x^2y^2 = 3$

$g'_x = 2x + 2xy^2 = 0 \qquad 2x(1+y^2) = 0 \Rightarrow x = 0$

$g'_y = 2y + x^2 \cdot 2y = 0 \qquad 2y(1+x^2) = 0 \Rightarrow y = 0$

$\Big\}$ $(0,0)$ <u>not adm</u> $\Downarrow$ no pts.

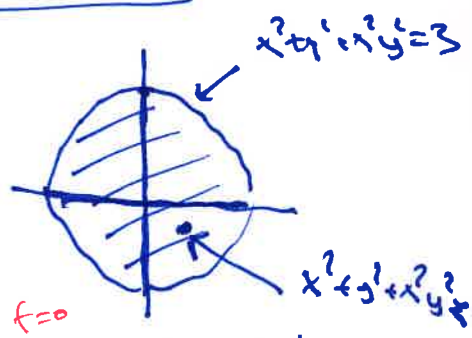
<u>Concl:</u> $f_{max} = \underline{\underline{1}}$ at $(x,y) = (\pm 1, \pm 1)$ with $\lambda = \underline{1/2}$

(2) <u>Kuhn-Tucker problems</u> : Closed inequality constraints ($\leq$ $\geq$)

<u>Ex:</u> $\boxed{\max\ f(x,y) = x^2y^2 \quad \text{when} \quad x^2+y^2+x^2y^2 \leq 3}$

<u>Cand pts:</u> - <u>boundary</u> : Lagrange (see comp. above)

- <u>interior</u>: Stationary pt. of f



$f'_x = 2xy^2 = 0 \qquad x = 0$

$f'_y = x^2 2y = 0 \qquad$ or $y = 0$

$(0, y)$ $-\sqrt{3} \leq y < \sqrt{3}$ $f = 0$

$(x, 0)$ $-\sqrt{3} < x < \sqrt{3}$ $f = 0$

compact $\Rightarrow$ there is a max

$\Rightarrow f_{max} = \underline{\underline{1}}$