Plan

1 Review
2 Exam 06/2021 Question 1-7

Remember: Course evaluation

About the exam:

May 23, 09-14 school exam { BI calculator, formulas with the exam

Tutorial May 20, 13- DI-065

15 questions + extra credit q.
- functions in one variable and derivatives
- integration
- matrices and vectors
- functions of two variables

Exam 06/2021: Home exam

1. $f(x) = 2\sqrt{x}\ln x - 4\sqrt{x} = 2x^{1/2}\ln x - 4x^{1/2}$

a) $f'(x) = 2\cdot\frac{1}{2}x^{-1/2}\cdot\ln x + 2x^{1/2}\cdot\frac{1}{x} - 4\frac{1}{2}x^{-1/2}$

$= \frac{\ln x}{\sqrt{x}} + \frac{2\cancel{\sqrt{x}}}{\cancel{x}\sqrt{x}} - \frac{2}{\sqrt{x}} = \frac{\ln x \cancel{+2} \cancel{-2}}{\sqrt{x}}$

$= \frac{\ln x}{\sqrt{x}}$ $(c=1, d=0)$

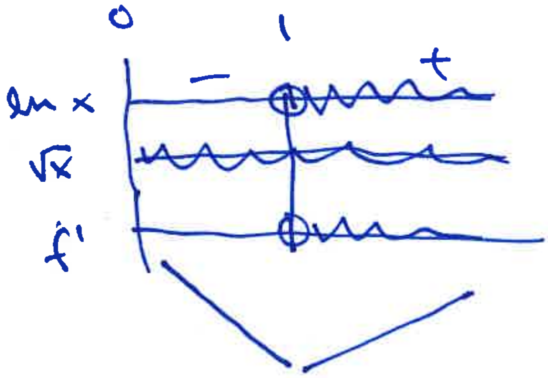
b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2\sqrt{x}(\ln x - 2) = \underline{\infty}$ (both factors $\to \infty$)

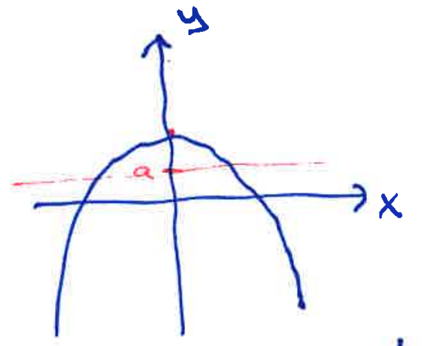
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2\sqrt{x}(\ln x - 2) = \lim_{x\to 0^+} \frac{\ln x - 2}{\frac{1}{2\sqrt{x}}}$ ($2\sqrt{x} \to 0$, $\ln x - 2 \to -\infty$)

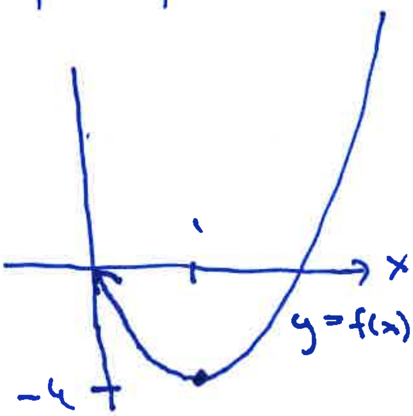
l'Hop.
$= \lim_{x\to 0^+} \frac{1/x}{1/2\cdot(-1/2)x^{-3/2}} = \lim_{x\to 0^+} \frac{-4x^{-1}x^{3/2}}{1} = \lim_{x\to 0^+} -4\sqrt{x} = \underline{0}$

c) Solutions of $f(x)=a$:

$$f'(x) = \frac{\ln x}{\sqrt{x}}, \quad x>0$$







$$f(x) = 2\sqrt{x}\,(\ln x - 2), \quad x>0$$

$f(x)=a$ has
$$\begin{cases} \text{no solutions if } a<-4 \\ \text{one " if } a=-4 \\ \text{two " if } -4<a<0 \\ \text{one " if } a\geq 0 \end{cases}$$

2. a) $$\int \frac{3-7x}{9-x^2}\,dx = \int \frac{-3}{3-x} + \frac{4}{3+x}\,dx = +3\ln|3-x| + 4\cdot\ln|3+x| + C$$

Partial fractions:

$$\frac{3-7x}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$$

$$3-7x = A(3+x) + B(3-x)$$

$$3-7x = \underbrace{(3A+3B)}_{=3} + \underbrace{(A-B)}_{=-7}x$$

$3(B-7)+3B=3$ $\qquad A=B-7$

$\frac{6B}{6}=\frac{24}{6}$ $\qquad A=-3$

$B=4$

b) $\int 15x\sqrt{x+1}\,dx = \int 15x\sqrt{u}\,du = \int 15(u-1)\sqrt{u}\,du$

$\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}} \to x = u-1$

$= \int 15\underbrace{u\sqrt{u}}_{u^{3/2}} - 15\underbrace{\sqrt{u}}_{u^{1/2}}\,du = 15\cdot\frac{2}{5}u^{5/2} - 15\cdot\frac{2}{3}u^{3/2} + C$

$= 6u^2\sqrt{u} - 10u\sqrt{u} + C = 6(x+1)^2\sqrt{x+1} - 10\sqrt{x+1}\cdot(x+1) + C$

c) $\int \frac{3\sqrt{\ln x}}{x}\,dx = \int \frac{3\sqrt{u}}{\cancel{x}}\cdot\cancel{x}\cdot du = \int 3u^{1/2}\,du$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$

$= \cancel{3}\cdot\frac{2}{\cancel{3}}u^{3/2} + C = 2u\sqrt{u} + C = 2\ln x\sqrt{\ln x} + C$

4. $A = \begin{pmatrix} 3 & 4 & 5 \\ 7 & 2 & 11 \\ 5 & 1 & 6 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$

a) $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 7 & 2 & 11 \\ 5 & 1 & 6 \end{vmatrix} = 3\cdot(12-11) - 4\cdot(42-55) + 5\cdot(7-10)$

$= 3\cdot 1 - 4\cdot(-13) + 5\cdot(-3)$

$= 3 + 52 - 15 = \underline{40} \neq 0$

$A^{-1} = \frac{1}{|A|}\cdot\underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots \\ \vdots & \cdots & \vdots \end{pmatrix}^T}_{\text{adj}(A)} = \frac{1}{40}\begin{pmatrix} 1 & 13 & -3 \\ -19 & -7 & 17 \\ 34 & 2 & -22 \end{pmatrix}^T = \frac{1}{40}\begin{pmatrix} 1 & -19 & 34 \\ 13 & -7 & 2 \\ -3 & 17 & -22 \end{pmatrix}$

b) $\left(\begin{array}{ccc|c} 3 & 4 & 5 & 24 \\ 7 & 2 & 11 & -20 \\ 5 & 1 & 6 & -6 \end{array}\right) \longleftarrow \quad A\cdot \underline{x} = \underline{b} \quad$ with $\underline{b} = \begin{pmatrix} 24 \\ -20 \\ -6 \end{pmatrix}$

Gauss

$A^{-1}\cdot |$

$$\underline{x} = A^{-1}\cdot \underline{b} = \frac{1}{40}\begin{pmatrix} 1 & -19 & 34 \\ 13 & -7 & 2 \\ -3 & 17 & -22 \end{pmatrix}\begin{pmatrix} 24 \\ -20 \\ -6 \end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix} 1\cdot 24 + 19\cdot 20 - 34\cdot 6 \\ 13\cdot 24 + 7\cdot 20 - 2\cdot 6 \\ -3\cdot 24 - 17\cdot 20 + 22\cdot 6 \end{pmatrix}$$

$$\underline{x} = \underline{\underline{\begin{pmatrix} 5 \\ 11 \\ -7 \end{pmatrix}}}$$

c) $\boxed{\begin{array}{l} A\underline{x} = b \\ x+y+z=9 \end{array}}$ $\quad \underline{b} = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 3 & 4 & 5 & r \\ 7 & 2 & 11 & s \\ 5 & 1 & 6 & t \end{array}\right) \begin{array}{l} -3 \\ -7 \\ -5 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & r-27 \\ 0 & -5 & 4 & s-63 \\ 0 & -4 & 1 & t-45 \end{array}\right) \begin{array}{l} 5 \\ 4 \\ -\frac{9}{14} \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & r-27 \\ 0 & 0 & 14 & s-63+5(r-27) \\ 0 & 0 & 9 & t-45+4(r-27) \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & * \\ 0 & 1 & 2 & * \\ 0 & 0 & 14 & * \\ 0 & 0 & 0 & ** \end{array}\right)$$

$(**) = 0$: one solution

$(**) \neq 0$: no solutions

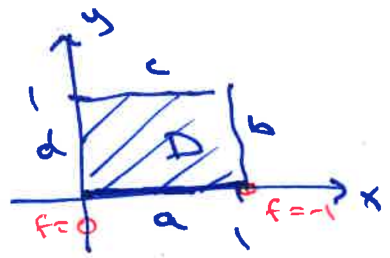
$$14\cdot | \quad ** = (t - 45 + 4r - 108) - \frac{9}{14}(s - 63 + 5r - 135) = 0$$

$$(\underline{14t} + \underline{56r} - 153\cdot 14) - (\underline{9s} + \underline{45r} - 135\cdot 9) = 0$$

$$11r - 9s + 14t = -135\cdot 9 + 153\cdot 14 = 360$$

$$\underline{\underline{11r - 9s + 14t = 360}} \longleftarrow$$ all vectors that satisfy this equation

5. max/min $f = \sqrt{xy} - x$ when $0 \le x, y \le 1$



Candidates:

- exterior pts of D that are stationary:

$f(x,y) = x^{1/2} y^{1/2} - x$

$f'_x = \frac{1}{2} x^{-1/2} \cdot y^{1/2} - 1 = 0$ $\quad \frac{\sqrt{y}}{2\sqrt{x}} - 1 = 0$

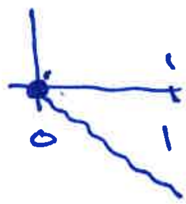
$f'_y = x^{1/2} \frac{1}{2} y^{-1/2} = 0$ $\quad \frac{\sqrt{x}}{2\sqrt{y}} = 0 \Rightarrow x = 0$

compact $\Rightarrow$ there is a max/min

not interior pt.

- boundary pts: (a) $y=0,\ 0 \le x \le 1$: $f(x,0) = -x$

max: $f(0,0) = 0$
min: $f(1,0) = -1$



(b) – –

(c) – –

(d) – –

$f_{max} = 1/4$ $\qquad$ $f_{min} = -1$

6. max $f = x + y$ when $\underbrace{y^2 = 5x^2 - x^3}_{C}$

a) Find $(x,y) \neq (0,0)$ on C s.t. $y' = -1$

$\boxed{y' = -\frac{f'_x}{f'_y}}$

~~derivative of~~ slope of the tangent to a point on the curve $\boxed{f(x,y) = c}$

$$y^2 = 5x^2 - x^3$$
$$0 = 5x^2 - x^3 - y^2$$
$$-1 = + \frac{10x - 3x^2}{+2y} \quad | \cdot 2y$$
$$\frac{-2y}{-2} = \frac{10x - 3x^2}{-2}$$
$$y = -5x + \frac{3}{2}x^2 = \frac{x}{2}(3x - 10)$$

$$y^2 = 5x^2 - x^3$$
$$\left(\frac{x}{2}(3x-10)\right)^2 = 5x^2 - x^3$$
$$4 \cdot | \quad \frac{x^2}{4}(9x^2 - 60x + 100) = 5x^2 - x^3 = x^2(5-x)$$
$$x^2(9x^2 - 60x + 100) = 4x^2(5-x)$$
$$9x^2 - 60x + 100 = 4(5-x) \quad \text{or } x = 0$$
$$9x^2 - 56x + 80 = 0 \qquad y = 0$$
$$x = \frac{56 \pm \sqrt{56^2 - 4 \cdot 9 \cdot 80}}{2 \cdot 9} \qquad \Downarrow$$
$$= \frac{28 \pm 8}{9} = 4, \ 20/9 \qquad (0,0)$$

Points:

$x = 4$, $y = 4$

$\Rightarrow$ $(4,4)$

$x = \frac{20}{9}$, $y = \frac{10}{9}\left(\frac{60}{9} - \frac{90}{9}\right)$

$= \frac{10 \cdot (-30)}{9 \cdot 9}$

~~not possible~~

$(20/9, -100/27)$

$\Downarrow$

$(4,4), \ (20/9, -100/27)$

b) max $f = x + y$ when $\underbrace{0}_{a} = \underbrace{5x^2 - x^3 - y^2}_{g(x,y)}$

$L = x + y - \lambda(5x^2 - x^3 - y^2)$

$$\boxed{\begin{aligned} L'_x &= 1 - \lambda\cdot(10x - 3x^2) = 0 \\ L'_y &= 1 - \lambda(-2y) = 0 \\ 5x^2 - x^3 &= y^2 \end{aligned}}$$

(1) $1 = \lambda\cdot(10x - 3x^2) \Rightarrow \lambda = \dfrac{1}{10x - 3x^2}$ $\quad 10x - 3x^2 \neq 0$

(2) $1 = -\lambda\cdot 2y \Rightarrow \lambda = \dfrac{1}{-2y}$ $\quad -2y \neq 0$

$\Downarrow$

$\lambda = \lambda$: $\dfrac{1}{10x - 3x^2} = \dfrac{1}{-2y}$ $\quad -2y = 10x - 3x^2$

$y = \dfrac{10x - 3x^2}{-2} = \dfrac{x(10 - 3x)}{-2}$

$\underline{y = \dfrac{x}{2}\cdot(3x - 10)}$

(3): $y^2 = 5x^2 - x^3$

$\left(\dfrac{x}{2}\right)^2 (3x - 10)^2 = 5x^2 - x^3 \xRightarrow{(a)}$ $(x,y) = (0,0)$ or $(x,y) = (4, \pm 2)$

$\Downarrow$

(0,0) not possible bc. (1), (2) (no λ)

~~$x = 4,\ y = \pm 2$~~

~~$\lambda = \frac{1}{40-48} = -1/8$~~

$(x,y) = (4,4)$: $\lambda = -1/8$

$(x,y) = \left(\frac{30}{9}, \frac{-100}{27}\right)$ $\quad \lambda = \dfrac{1}{2\cdot 100/27} = \underline{\dfrac{27}{200}}$

Candidates: $(4, 4; -1/8)$, $(20/9, -100/27; 27/200)$

3. a) Asympt.: $x = 1.25$, $y = 1$ $\qquad f'(x) = 1 + \frac{c}{x - 1.25}$

From graph: Read off $(1, 3)$: $3 = 1 + \frac{c}{1 - 1.25}$

$$2 = \frac{c}{-0.25} = \frac{4c}{-1}$$

$$4c = -2 \quad c = -1/2$$

$$f'(x) = 1 + \frac{-1/2 \cdot 4}{x - 5/4 \cdot 4}$$

$$= 1 + \frac{-2}{4x-5} = \frac{4x-5-2}{4x-5}$$

$$f'(x) = \frac{4x-3}{4x-5}$$

b) $f(3) - f(2) = \int_2^3 f'(x)\,dx$ ← since $f(x)$ is an antiderivative of $f'(x)$ by defn.

Read off area under the graph of $f'(x)$ in the figure in $[2, 3]$ $\simeq 9 \text{ sq} \cdot \left(\frac{1}{4} \cdot \frac{1}{4}\right)$ $= 9/16$

c) $f(3) - f(2) = \int_2^3 1 - \frac{2}{4x-5}\,dx$

$$= \left[x - \tfrac{1}{2}\ln(4x-5)\right]_2^3 = \ldots = 1 - \tfrac{1}{2}\ln(7/3)$$

Extra Credit Q: 6c, 7 See solutions