EBA2911, lecture 4, 8 Sept. 2021, Runar Ile

Plan:
1. Regular cash flows
2. Infinite series and limit values
3. Euler's number and continuous compounding

1. Regular cash flows

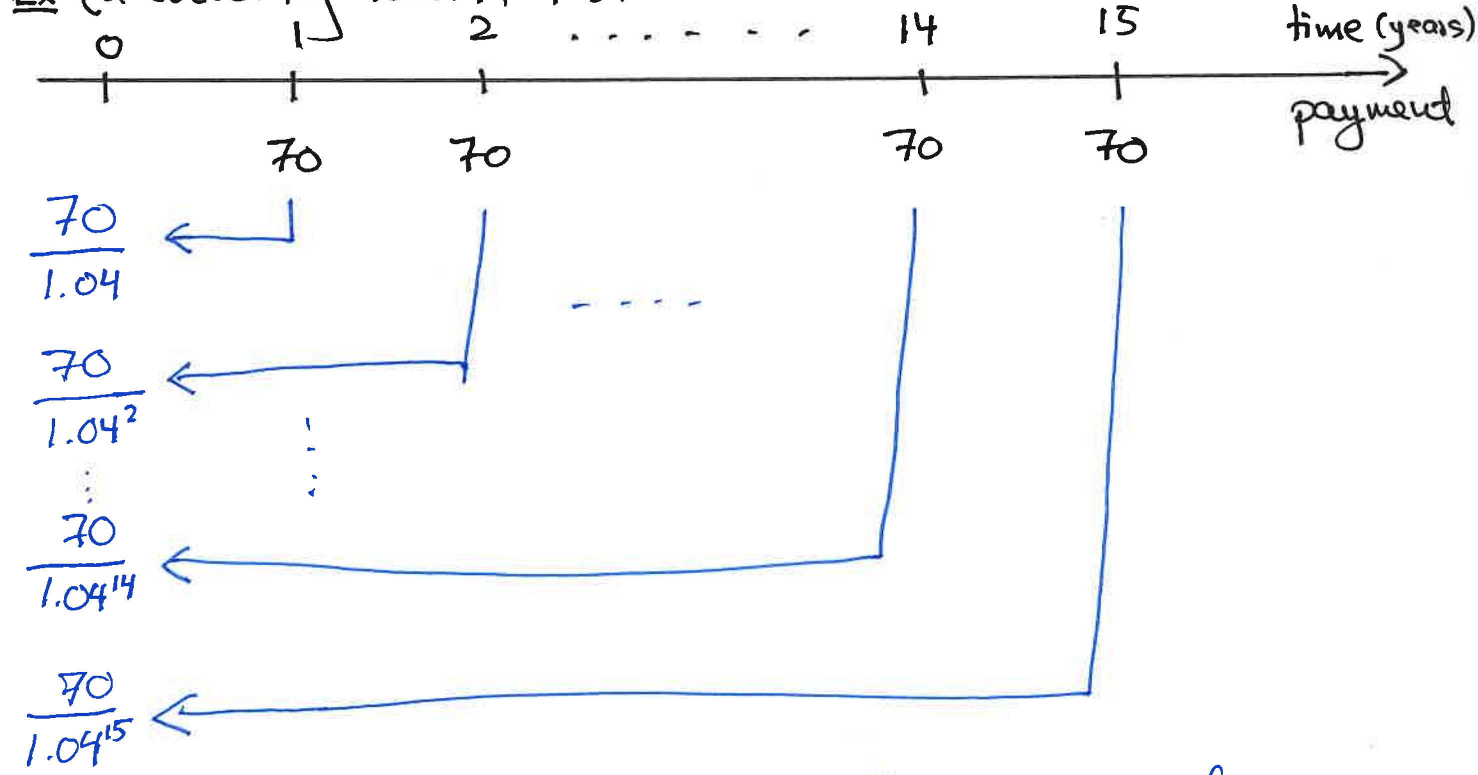
A fixed amount is paid every period.

Ex Annuity loan (tot. present value = what you borrow)

Ex Saving with a fixed amount each period.
Future value: The balance (what you have saved)

- both are geometric series.

Ex (annuity loan, 4%)



The sum is the tot. present value of the cash flow = what you can borrow

We get a geometric series:

$$\frac{70}{1.04} + \frac{70}{1.04^2} + \ldots + \frac{70}{1.04^{14}} + \frac{70}{1.04^{15}}$$

We can read the series backwards:

$$a_1 = \frac{70}{1.04^{15}}, \quad k = 1.04, \quad n = 15$$

Then the sum (tot. pres. value) is

$$a_1 \cdot \frac{k^n - 1}{k - 1} = \frac{70}{1.04^{15}} \cdot \frac{1.04^{15} - 1}{0.04} = \underline{\underline{778.29}}$$

We could read the sum forwards

Then $a_1 = \frac{70}{1.04}$, $k = \frac{1}{1.04}$, $n = 15$ so

the sum becomes $\frac{70}{1.04} \cdot \frac{\left(\frac{1}{1.04}\right)^{15} - 1}{\frac{1}{1.04} - 1} = \underline{\underline{778.29}}$

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2. Infinite series and limit values.

Ex The annuity: 70 000

interest: 4 %

number of years: n

First payment: One year from now

The total present value:

$$\frac{70'}{1.04^n} \cdot \frac{1.04^n - 1}{0.04} = \frac{70' \cdot (1.04^n - 1)}{1.04^n \cdot 0.04}$$

$$= \frac{70' \cdot (1.04^n - 1) : 1.04^n}{\cancel{1.04^n} \cdot 0.04 : \cancel{1.04^n}} = \frac{70' \cdot \left(\frac{1.04^n}{1.04^n} \overset{1}{} - \frac{1}{1.04^n} \right)}{0.04}$$

approaches 0 when $n \to \infty$ "goes to infinity"

So the tot. pres. value is approaching

$$\frac{70'}{0.04} \quad \text{when } n \to \infty$$

$$= \underline{\underline{1\,750\,000}}$$

<u>Conclusion</u> If you pay the bank 70 000 each year forever, starting next year and the interest is 4% the you can borrow $\frac{70\,000}{0.04} = 1\,750\,000$

3. Euler's number and continuous compounding

Ex You deposit 1000 into an account with 12% nominal interest.

compounding	balance after 1 year
Annual	$1000 \cdot 1.12 = 1120.00$
Half year	$1000 \cdot 1.06^2 = 1123.60$
Quarterly	$1000 \cdot 1.03^4 = 1125.51$
Monthly	$1000 \cdot 1.01^{12} = 1126.83$
Daily	$1000 \cdot \left(1 + \frac{12\%}{365}\right)^{365} = 1127.47$

Pattern (n periods) $1000 \cdot \left(1 + \frac{0.12}{n}\right)^n$

!!

Euler's number: $e = 2.718281\ldots$

Calculator: 1 $\boxed{e^x}$

Calculate: $1000 \cdot e^{0.12} = 1127.50$

1000 $\boxed{\times}$ 0.12 $\boxed{e^x}$ $\boxed{=}$

Start: 9.00

Euler's number e is defined as the limit of $\left(1+\frac{1}{n}\right)^n$ when n approaches infinity (∞) ('becomes bigger and bigger')

Write: $\left(1+\frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} e$

$\underline{\text{Ex}}$ $\left(1+\frac{1}{1000}\right)^{1000} = 2.71692\ldots.$

$$\left(1+\frac{1}{1\,\text{mill}}\right)^{1\,\text{mill}} = 2.718280$$

Back to the example with 12%

$$\left(1+\frac{0.12}{n}\right)^n = \left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^n$$

$$= \Bigg[\underbrace{\left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{\frac{n}{0.12}}}_{\substack{\text{approaches } e \\ \text{as } n\to\infty}}\Bigg]^{0.12} \xrightarrow[n\to\infty]{} e^{0.12}$$

So $1000\cdot\left(1+\frac{0.12}{n}\right)^n \longrightarrow 1000\cdot e^{0.12}$

After 1 year with 12% nominal interest
and <u>continuous compounding</u>
the deposit of 1000 has increased to

$$1000\cdot e^{0.12} = 1127.50$$

the growth factor
for 1 year
with continuous compounding

Annual growth factor $e^{0.12} = 1.12750$

The <u>effective</u> interest is $e^{0.12} - 1 = 0.12750$

$= 12.750\%$

After 2 years with continuous compounding:

$$1000 \cdot e^{0.12} \cdot e^{0.12} = 1000 \cdot (e^{0.12})^2$$

$$= 1000 \cdot e^{0.12 \cdot 2} = 1000 \cdot e^{0.24}$$

$$= \underline{\underline{1271.25}}$$

<u>Problem</u> You deposit 10 mill. into an account with 2.8% nominal interest

Calculate the balance after 5 years with

a) Annual compounding

b) Continuous ——"——

c) Compute the effective (annual) interest with continuous compounding.

<u>Solution</u>

a) Annual growth factor : 1.028

Balance after 5 years: $10 \text{ mill} \cdot 1.028^5$

$= \underline{\underline{11.48 \text{ mill}}}$

b) Annual growth factor: $e^{0.028} = 1.0284$

Balance after 5 years: $10 \text{ mill} \cdot (e^{0.028})^5$

$= 10 \text{ mill} \cdot e^{0.028 \cdot 5}$

$= 10 \text{ mill} \cdot e^{0.140}$

$= \underline{\underline{11.50 \text{ mill}}}$

c) The effective interest is

$e^{0.028} - 1 = 1.0284 - 1 = 0.0284$

$= \underline{\underline{2.84\,\%}}$