EBA 1180, lecture 13, 11 Oct. 2022, Runar Ile

Plan
1. Rational functions and asymptotes
2. Hyperbolas

1. Rational functions and asymptotes

Rational function $f(x) = \frac{p(x)}{q(x)}$ ← polynomials

Ex $f(x) = \frac{2x+1}{x^2+3}$ - would like to see what happens when x is big

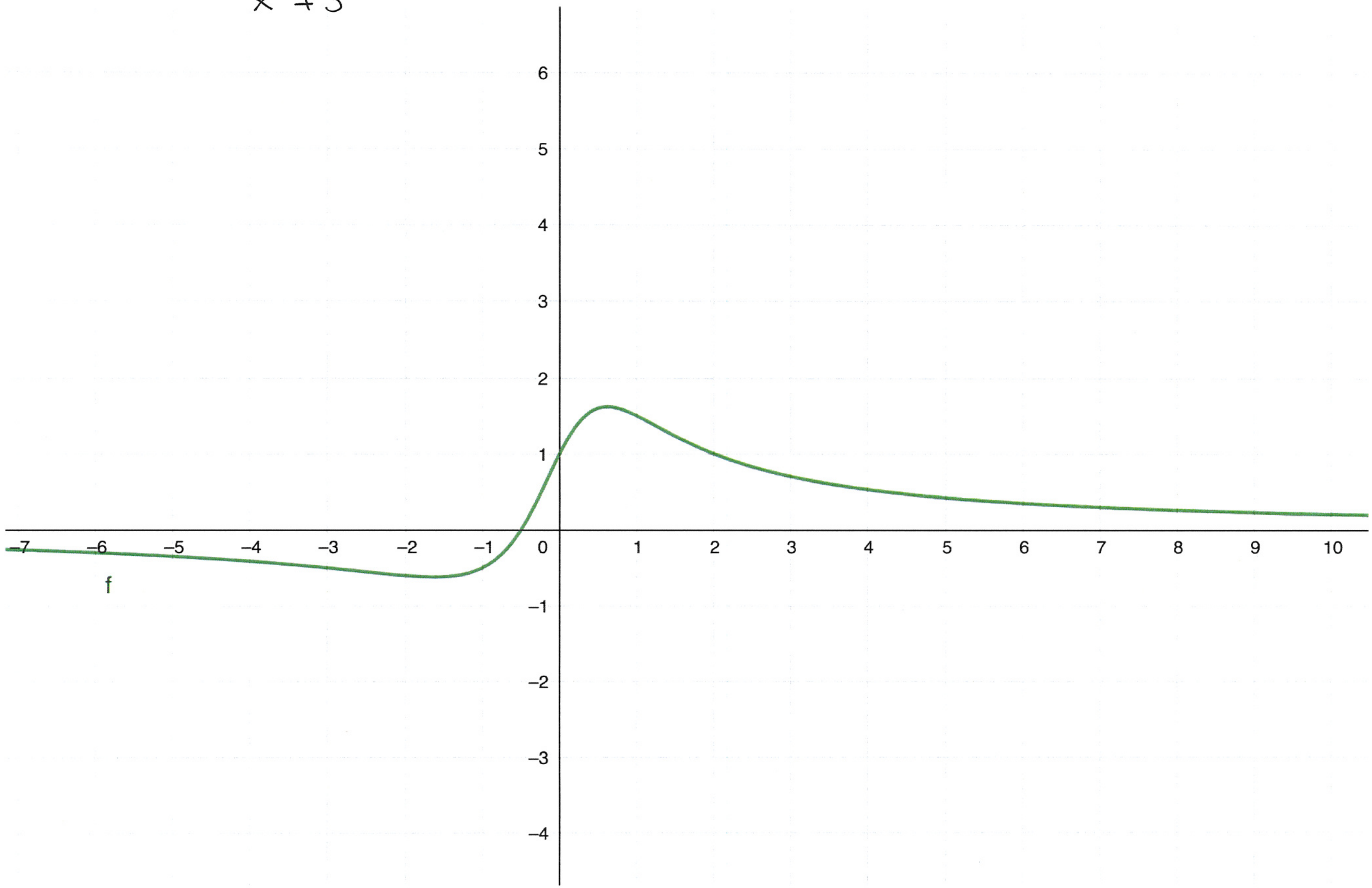
- divide by x^2 both in the numerator and in the denominator

$$= \frac{\frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} \xrightarrow[x \to \pm\infty]{} \frac{0}{1} = 0$$

$$f(1000) = \frac{\frac{2}{1000} + \frac{1}{1000^2}}{1 + \frac{3}{1000^2}} = 0.00200099\ldots$$

This means that the line $y=0$ (x free) is a horizontal asymptote for $f(x)$. So the graph of $f(x)$ is approaching the x-axis (the horizontal asymptote)

<u>Ex</u> $f(x) = \dfrac{2x+1}{x^2+3}$



EX $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$ $\quad (x \neq 1,\ x \neq 5)$

What happens when x is approaching 1 or 5 ?

If $x \to 1^-$ ("x is approaching 1 from below" $x = 0.9$, $x = 0.99$, $x = 0.999$ ...)

then

$$\left.\begin{aligned} x-1 &\longrightarrow 0^- \\ x-5 &\longrightarrow -4^- \\ 2x+1 &\longrightarrow 3^- \end{aligned}\right\} \text{implies } f(x) = \frac{\overbrace{(2x+1)}^{3^-}}{\underbrace{(x-1)}_{0^-}\underbrace{(x-5)}_{-4^-}} \xrightarrow[x \to 1^-]{} +\infty$$

If $x \to 1^+$ e.g. $x = 1.1$, 1.01, 1.001

then

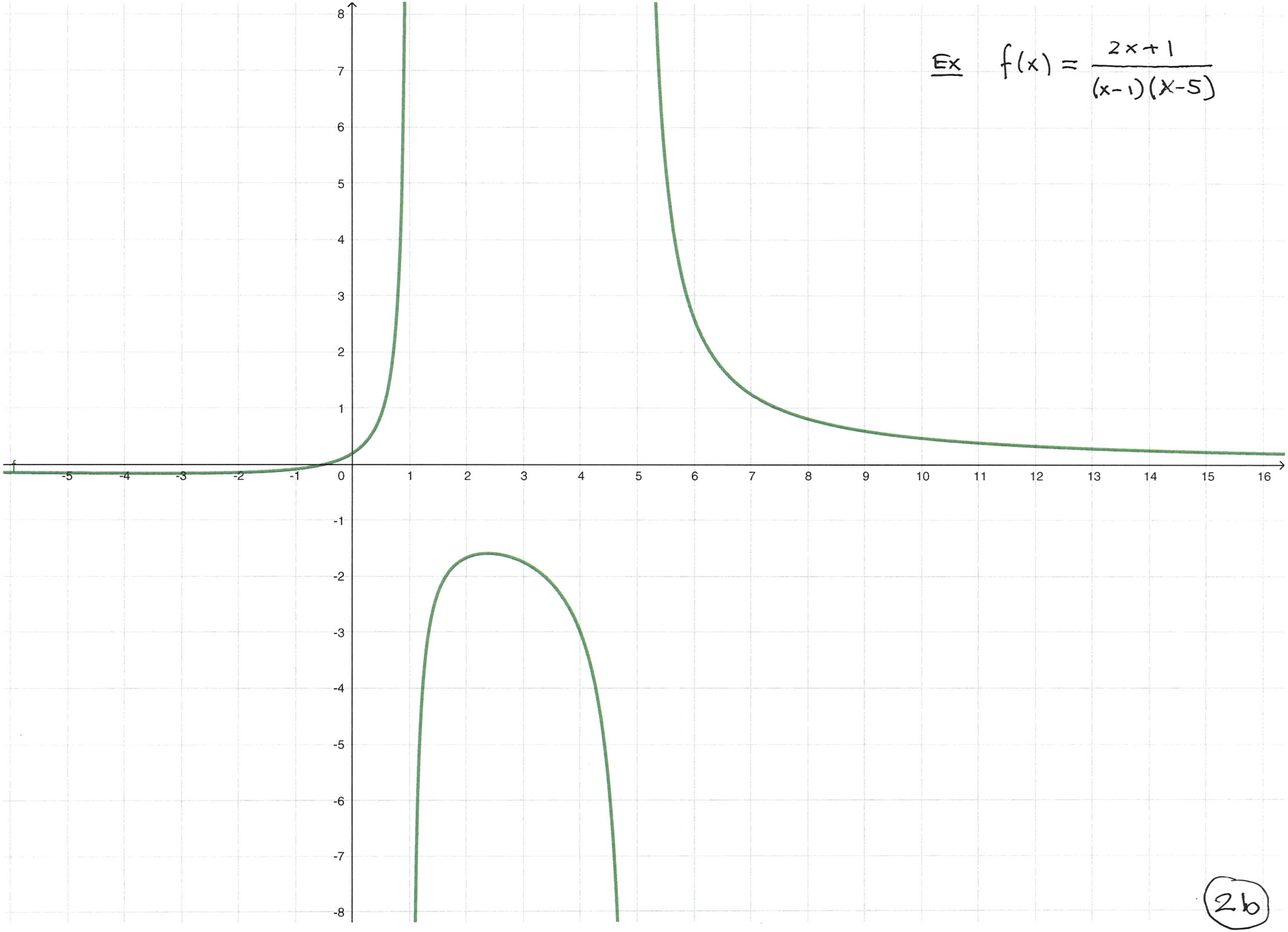
$$\left.\begin{aligned} x-1 &\longrightarrow 0^+ \\ x-5 &\longrightarrow -4^+ \\ 2x+1 &\longrightarrow 3^+ \end{aligned}\right\} \text{implies } f(x) = \frac{\overbrace{2x+1}^{3^+}}{\underbrace{(x-1)}_{0^+}\underbrace{(x-5)}_{-4^+}} \xrightarrow[x \to 1^+]{} -\infty$$

Conlusion The line $x = 1$ (y free) is a vertical asymptote for $f(x)$.

The line $x = 5$ (y free) is a vertical asymptote for $f(x)$: $\quad f(x) \xrightarrow[x \to 5^-]{} -\infty$, $\quad f(x) \xrightarrow[x \to 5^+]{} +\infty$

Note : $f(x)$ also has a horizontal asymptote $y = 0$ (the x-axis)

Ex $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$



2b

Non-vertical asymptotes

Ex $f(x) = x - 5 + \frac{2}{x-4}$ has a vertical asymptote $x=4$

Put $g(x) = x - 5$

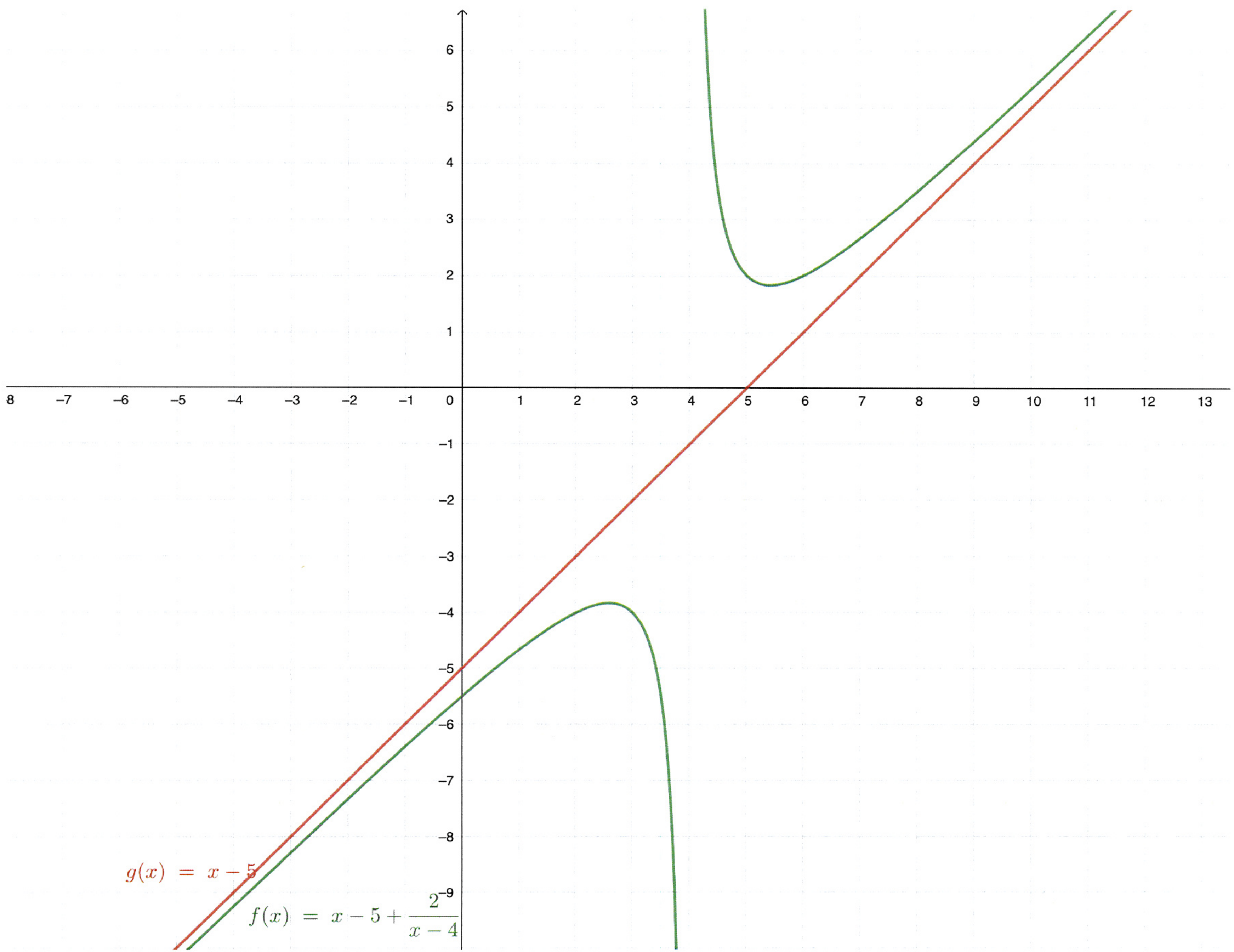
Then the graph of $f(x)$ is approaching the graph of $g(x)$ when $x \to \pm\infty$ because

$$f(x) - g(x) = \frac{2}{x-4} \xrightarrow[x \to \pm\infty]{} 0$$

Note that $f(x) = \frac{(x-5)(x-4)+2}{x-4} = \frac{x^2-9x+22}{x-4}$

- have to do polynomial division to find the better expression for $f(x)$

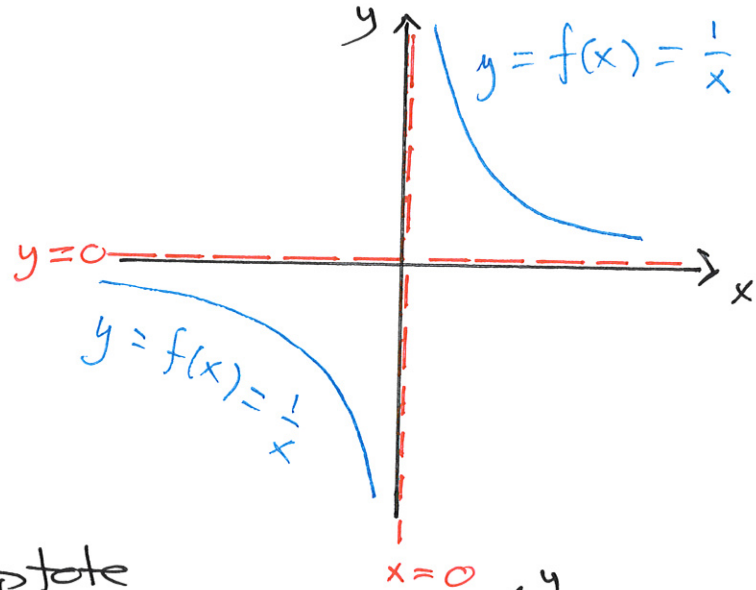
Start: 11.06



6
5
4
3
2
1
8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10 11 12 13
−1
−2
−3
−4
−5
−6
−7
−8
−9
$g(x) = x - 5$
$f(x) = x - 5 + \frac{2}{x-4}$

3b

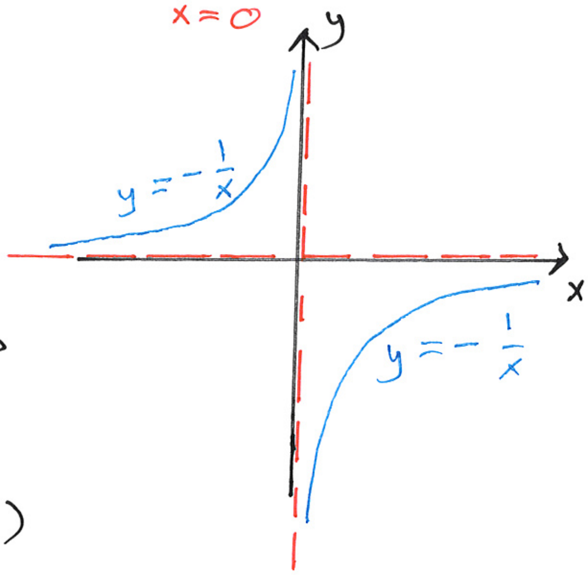
2. Hyperbolas



<u>Ex</u> $f(x) = \frac{1}{x} \quad (x \neq 0)$

The line $y = 0$ is a horizontal asymptote

The line $x = 0$ is a vertical asymptote



<u>Ex</u> $f(x) = -\frac{1}{x} \quad (x \neq 0)$

– has the same asymptotes

<u>Definition</u> A function $f(x)$ is a hyperbola function if it can be written as

$$f(x) = c + \frac{a}{x-b} \qquad (a \neq 0)$$

<u>Ex</u> $f(x) = \dfrac{3x-5}{x-2}$ is a hyperbola function, but not on standard form.

$$(3x-5) : (x-2) = 3 + \frac{1}{x-2} \quad \text{so} \quad a = 1,\; b = 2,\; c = 3$$

$$\begin{array}{r} -(3x-6) \\ \hline 1 \end{array} \qquad \cdot (x-2)$$

We have $3 + \frac{1}{x-2} \xrightarrow[x \to 2^-]{} -\infty$ and $3 + \frac{1}{x-2} \xrightarrow[x \to 2^+]{} +\infty$

So the line $x = 2$ is a vertical asymptote.

Also note that $3 + \dfrac{1}{x-2} \xrightarrow[x \to \pm\infty]{} 3^{\pm}$

so the line $y = 3$ is a horizontal asymptote.

$f(1) = 3 + \frac{1}{1-2} = 2$

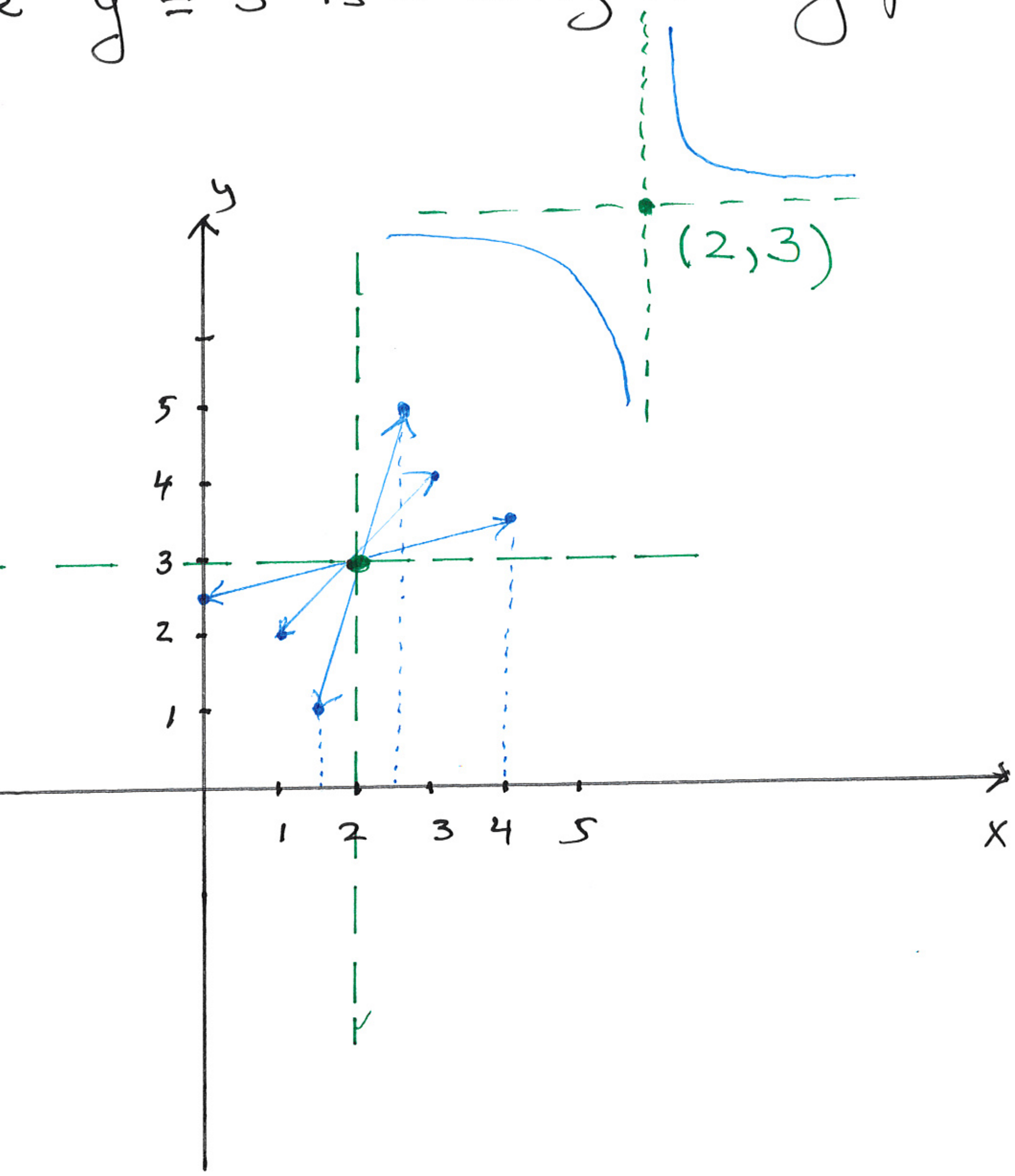
$f(3) = 3 + \frac{1}{3-2} = 4$

$f(1.5) = 3 + \frac{1}{1.5-2} = 1$

$f(2.5) = 3 + \frac{1}{2.5-2} = 5$

$f(0) = 3 + \frac{1}{0-2} = 2.5$

$f(4) = 3 + \frac{1}{4-2} = 3.5$



The graph of a hyperbola function is symmetric through the intersection point of the asymptotes.

2019s Multiple Choice

Problem 5

We have the hyperbola function $f(x) = \dfrac{4x-38}{x-10}$. Which of the graphs in figure 1 is the graph of $f(x)$?

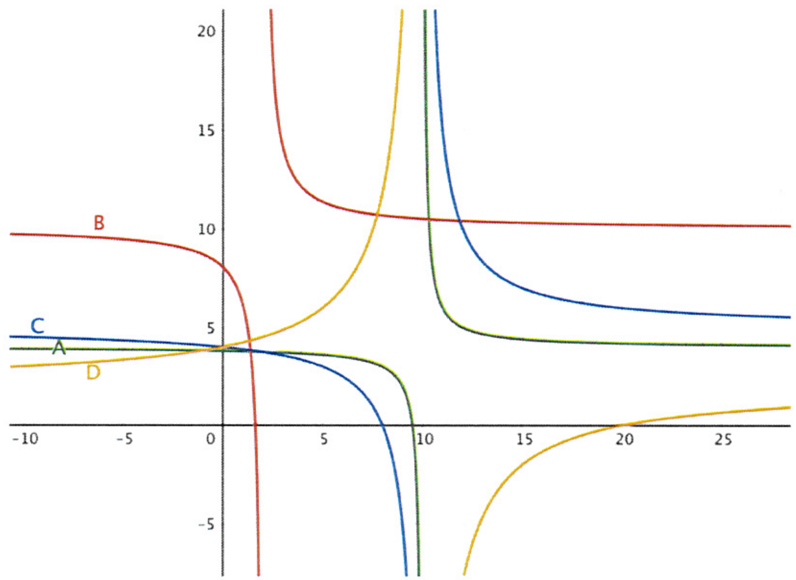


Figure 1: Graphs A-D

(A) $f(x)$ has the graph A (green)
(B) $f(x)$ has the graph B (red)
(C) $f(x)$ has the graph C (blue)
(D) $f(x)$ has the graph D (yellow)
(E) I choose not to answer this problem.

5b

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Find the expression for the hyperbola function.

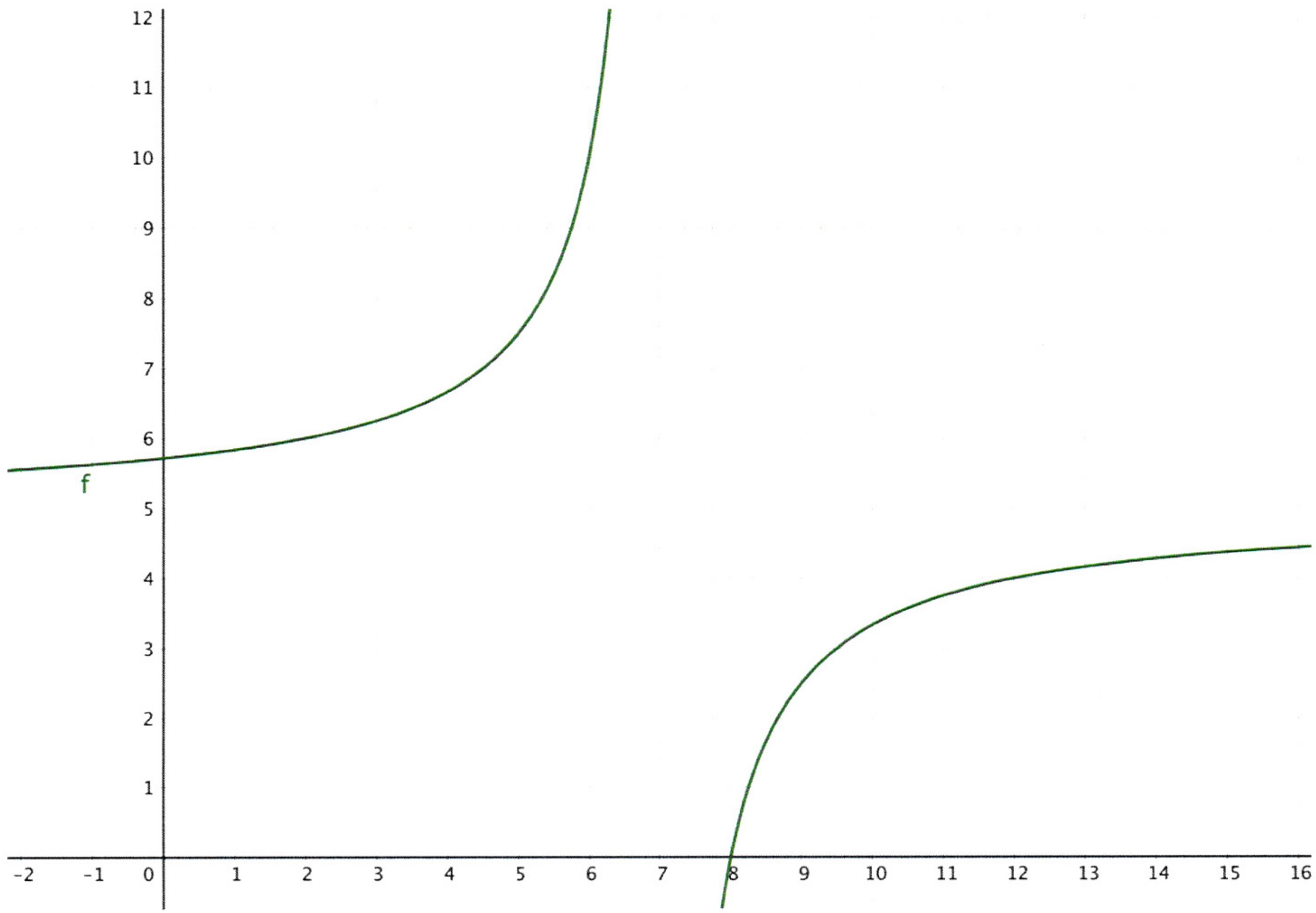

Figure 2: Hyperbola

5c