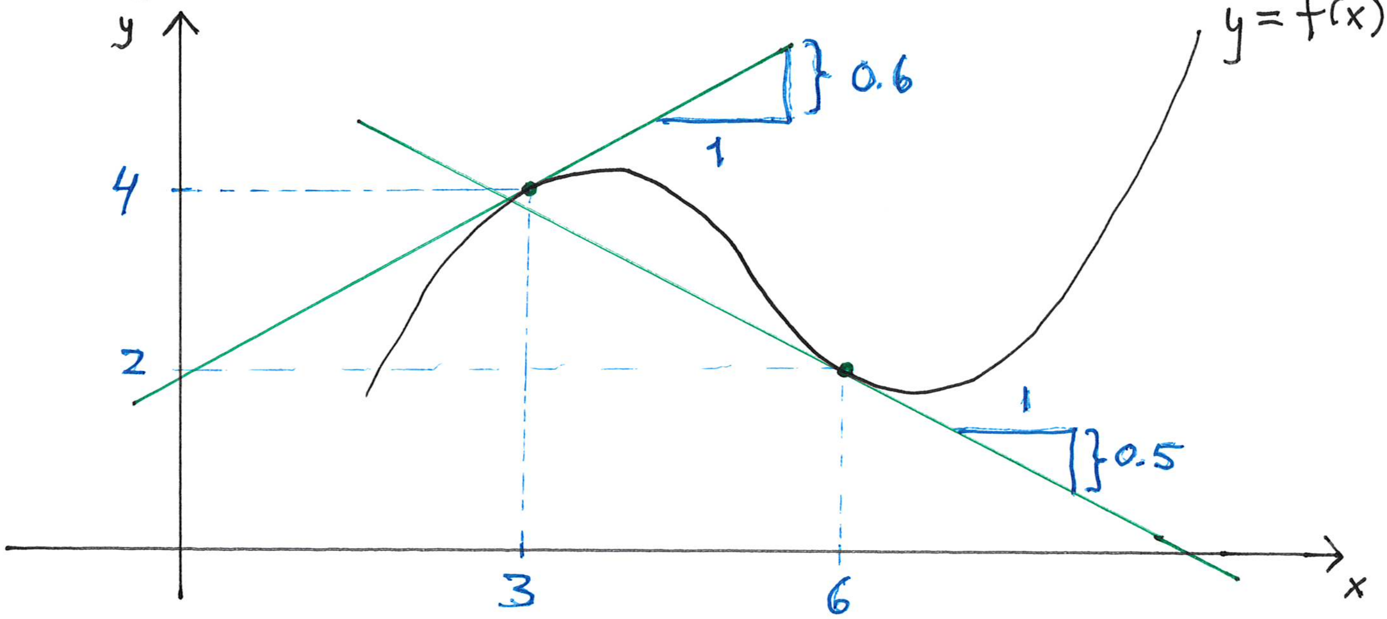
EBA 1180, lecture 16, 26 Oct. 2022, Runar Ile

Plan:
1. Tangents and the derivative
2. The derivative as a function
3. Rules for differentiation

1. Tangents and the derivative

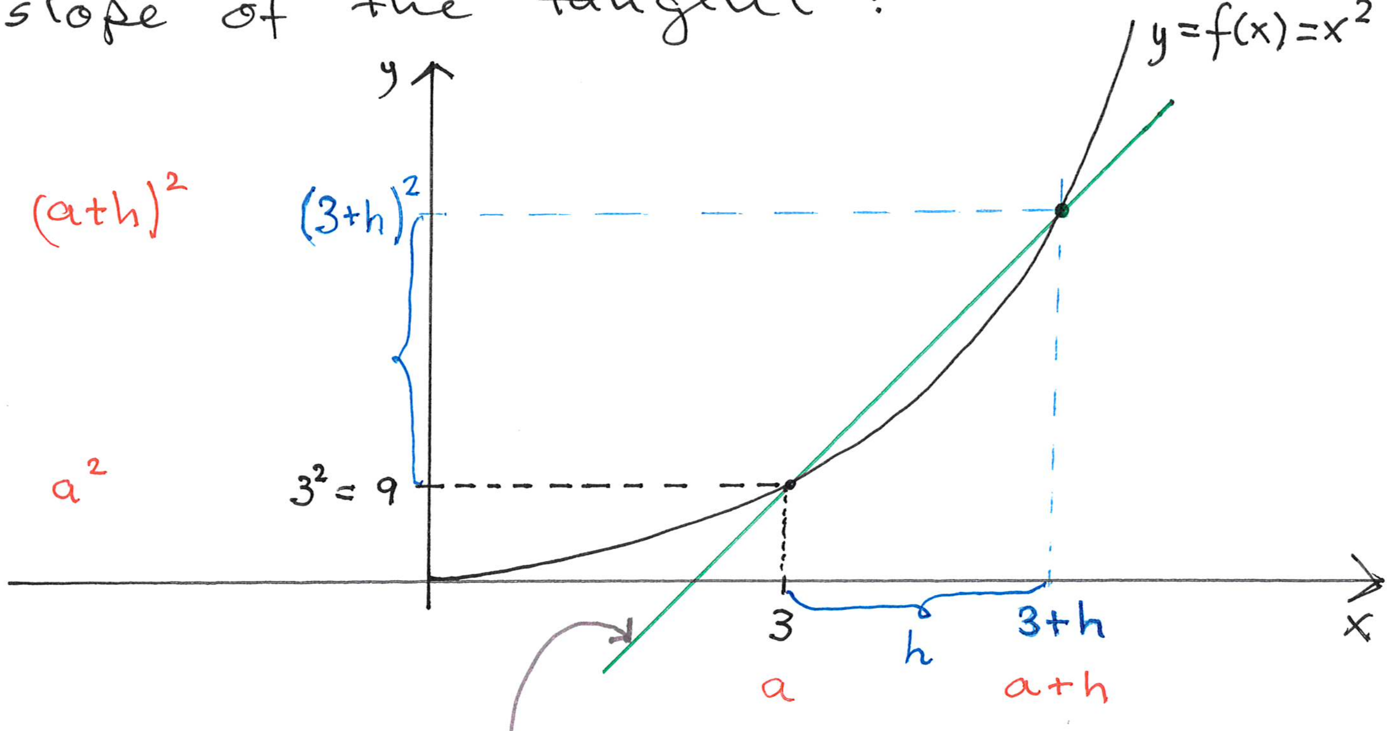


- The tangent of the graph of f(x) at the point (3, 4) has slope 0.6
  We write $f'(3) = 0.6$
- The tangent of the graph of f(x) at the point (6, 2) has slope -0.5
  We write $f'(6) = -0.5$

Two important applications
1) To determine where the function increases/decreases and max/min
2) Approximate complicated functions with linear functions
   - typical for economic models

How to find the slope of the tangent?

Ex $f(x) = x^2$ and $(3, 9)$. What is the slope of the tangent?



The slope of this secant line is

$$\frac{(a+h)^2 - a^2}{} \qquad \frac{(a+h)(a+h) - a^2}{}$$

$$\frac{\text{change in } y}{\text{change in } x} = \frac{(3+h)^2 - 3^2}{h} = \frac{(3+h)(3+h) - 9}{h}$$

$$a^2 + 2\cdot ah + h^2 - a^2 \qquad h^2 + 2ah \qquad h(h+2a)$$

$$= \frac{9 + 2\cdot 3h + h^2 - 9}{h} = \frac{h^2 + 6h}{h} = \frac{\cancel{h}(h+6)}{\cancel{h}}$$

$$h + 2a \xrightarrow[h \to 0]{} 2a$$

$= h + 6 \xrightarrow[h\to 0]{} 6$ which has to be the slope of the tangent line to $f(x)$ through $(3, 9)$.

We write $f'(3) = 6$

also $f'(a) = 2a$

Start: 11.00

2. The derivative as a function

In the example: If $x = a$ then $f'(a) = 2a$
- this is a function, we use x as variable:

$$f'(x) = 2x$$

E.g. the slope of tangent of $f(x)$
- at $(-3, 9)$ is $f'(-3) = 2\cdot(-3) = -6$
- at $(1, 1)$ is $f'(1) = 2\cdot 1 = 2$
- at $(10, 100)$ is $f'(10) = 2\cdot 10 = 20$

We could do a similar calculation with $f(x) = x^3$ and get $f'(x) = 3x^2$

3. Rules of differentiation

Power rule $\quad f(x) = x^n$ gives $f'(x) = n\cdot x^{n-1}$ for all n.

Ex $f(x) = x^{10}$, $\quad f'(x) = 10\cdot x^9 \quad (n = 10)$

Ex $f(x) = \sqrt[3]{x}$, $\quad f'(x) = \frac{1}{3}\cdot x^{\frac{1}{3}-1} \quad (n = \frac{1}{3})$

$= x^{\frac{1}{3}}$

$$= \frac{1}{3}\cdot x^{-\frac{2}{3}}$$

$$= \frac{1}{3}\cdot\frac{1}{x^{2/3}}$$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{x^2}}$$

__The sum rule__ If $f(x) = g(x) + h(x)$

then $f'(x) = g'(x) + h'(x)$

__Ex__ $f(x) = x + x^3$ then $f'(x) = 1 + 3x^2$

__The constant rule__ If k is a constant number

and $f(x) = k \cdot g(x)$ then

$f'(x) = k \cdot g'(x)$

__Ex__ $k = 7$, $g(x) = x^2$, then $f(x) = 7x^2$

and $f'(x) = 7 \cdot 2x = 14x$

__The product rule__ If $f(x) = g(x) \cdot h(x)$

then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

__Ex__ $f(x) = (5x^3 - 2x + 1) \cdot (3x + 7)$

Calculate $f'(x)$ by using the product rule.

Solution: $g(x) = 5x^3 - 2x + 1$ and $h(x) = 3x + 7$

$g'(x) = 15x^2 - 2$ $h'(x) = 3$

so $f'(x) = (15x^2 - 2) \cdot (3x + 7) + (5x^3 - 2x + 1) \cdot (3)$

note the parentheses!

calculate

$= \underline{\underline{60x^3 + 105x^2 - 12x - 11}}$

The quotient rule

Suppose $f(x) = \dfrac{g(x)}{h(x)}$

Then

$$f'(x) = \frac{g'(x)\cdot h(x) - g(x)\cdot h'(x)}{[h(x)]^2}$$

<u>Ex</u> $f(x) = \dfrac{3x+1}{2x+5}$. Then

$g(x) = 3x+1$ and $h(x) = 2x+5$

$g'(x) = 3$ $\qquad$ $h'(x) = 2$

note the parentheses !

$$f'(x) = \frac{3\cdot(2x+5) - (3x+1)\cdot 2}{(2x+5)^2}$$

note the sign − for the whole parenthesis!

$$= \frac{3\cdot 2x + 3\cdot 5 - (3x\cdot 2 + 1\cdot 2)}{(2x+5)^2}$$

note this − !!!

$$= \frac{6x + 15 - 6x - 2}{(2x+5)^2}$$

$$= \frac{13}{(2x+5)^2}$$

usually better not to expand the denominator.

The chain rule

the inner function

If $f(x) = g(\underbrace{u(x)})$

the outer function

then $f'(x) = g'(u) \cdot u'(x)$ where $u = u(x)$

Ex $f(x) = (x^2+2)^{10}$

Put $u = u(x) = x^2+2$ and $g(u) = u^{10}$

$u'(x) = 2x$ $\quad g'(u) = 10u^9$

Then $f'(x) = 10u^9 \cdot 2x$

$= 10(x^2+2)^9 \cdot 2x = \underline{\underline{20x(x^2+2)^9}}$

Two functions

$f(x) = e^x$ and $g(x) = \ln(x)$

$f'(x) = e^x$ $\quad g'(x) = \frac{1}{x}$

Ex $f(x) = e^{3x}$

$u(x) = 3x$, $g(u) = e^u$

$u'(x) = 3$ $\quad g'(u) = e^u$

$f'(x) = e^u \cdot 3$

$= \underline{\underline{3e^{3x}}}$

Ex $f(x) = \ln(x^2+1)$

$u(x) = x^2+1$, $g(u) = \ln(u)$

$u' = 2x$ $\quad g'(u) = \frac{1}{u}$

$f'(x) = \frac{1}{u} \cdot 2x$

$= \underline{\underline{\frac{2x}{x^2+1}}}$